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THE TYPE A UNCERTAINTY

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FORMULATION OF THE PROBLEM

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FORMULATION OF THE PROBLEM

The following model is considered

$$\begin{pmatrix} \widehat{\mathbf{\Theta}} \\ \mathbf{Y} \end{pmatrix} \sim N_{l+n}\left[\begin{pmatrix} \mathbf{I}, & \mathbf{0} \\ \mathbf{D}, & \mathbf{X} \end{pmatrix}\begin{pmatrix} \mathbf{\Theta} \\ \boldsymbol{\beta} \end{pmatrix}, \begin{pmatrix} \mathbf{W}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{\Sigma} \end{pmatrix}\right]$$

$\mathbf{\Theta}$ ... **the parameter of the 1st stage,**
$\boldsymbol{\beta}$ ... **the parameter of the 2nd stage,**
$\mathbf{W}$ ... **the type B uncertainty.**

The BLUE of the parameter $\boldsymbol{\beta}$ is

$$\widehat{\boldsymbol{\beta}}(\mathbf{Y}, \widehat{\mathbf{\Theta}}) = [\mathbf{X}'(\mathbf{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{X}]^{-1}\mathbf{X}'(\mathbf{\Sigma} + \mathbf{DWD}')^{-1}(\mathbf{Y} - \mathbf{D}\widehat{\mathbf{\Theta}})$$

and its covariance matrix

$$\begin{aligned} \mathrm{Var}[\widehat{\boldsymbol{\beta}}(\mathbf{Y}, \widehat{\mathbf{\Theta}})] &= [\mathbf{X}'(\mathbf{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{X}]^{-1} \\ &= (\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1} + (\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{DWD}'\mathbf{\Sigma}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}. \end{aligned}$$

The type A uncertainty= $(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}$.

The type B uncertainty= $(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{DWD}'\mathbf{\Sigma}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}$.

If $\mathbf{\Sigma} = \sigma^2\mathbf{V}$ **the problem is to estimate** σ^2 **on the basis** $\mathbf{Y}$ **resp.** $\mathbf{Y} - \mathbf{D}\widehat{\mathbf{\Theta}}$.

Assumption: the model is regular and W is known matrix.

ESTIMATION OF THE TYPE A UNCERTAINTY IN LINEAR MODELS

<u>**The estimator based on Y :**</u>

$$\widehat{\sigma}_1^2 = \mathbf{Y}'\Big(\mathbf{M}_{(D,X)}\mathbf{V}\mathbf{M}_{(D,X)}\Big)^+\mathbf{Y}/[n - r(\mathbf{D},\mathbf{X})]$$

$$\mathrm{Var}(\widehat{\sigma}_1^2) = \frac{2\sigma^4}{n - r(\mathbf{D},\mathbf{X})}.$$

need not exist ! $(n = r(\mathbf{D},\mathbf{X}))$

<u>**The estimator based on $\mathbf{Y} - \mathbf{D}\widehat{\boldsymbol{\Theta}}$:**</u>

$$\widehat{\sigma}_2^2 = \frac{A - B}{\mathrm{Tr}[(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}]}.$$

$$\begin{aligned}
\boldsymbol{\Sigma}_0 &= \sigma_0^2\mathbf{V} + \mathbf{DWD}', \\
A &= (\mathbf{Y} - \mathbf{D}\widehat{\boldsymbol{\Theta}})'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+(\mathbf{Y} - \mathbf{D}\widehat{\boldsymbol{\Theta}}), \\
B &= \mathrm{Tr}[(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{DWD}']
\end{aligned}$$

$$\mathrm{Var}_{\sigma_0^2}(\widehat{\sigma}_2^2) = \frac{2}{\mathrm{Tr}[(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}]}.$$

<u>**The estimator based on $\mathbf{Y} - \mathbf{D}\widehat{\boldsymbol{\Theta}}$:**</u>

$$\widehat{\sigma}_3^2 = (\mathbf{Y} - \mathbf{D}\widehat{\boldsymbol{\Theta}})'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+(\lambda_1\mathbf{V} + \lambda_2\mathbf{DWD}')(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+(\mathbf{Y} - \mathbf{D}\widehat{\boldsymbol{\Theta}})$$

$$\mathrm{Var}_{\sigma_0^2}(\widehat{\sigma}_3^2) = 2(1,0)\mathbf{S}_{V,DWD'}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where

$$\mathbf{S}_{V,DWD'}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\begin{aligned}
\mathbf{S}_{V,DWD'} &= \begin{pmatrix} \boxed{\mathbf{aa}}, & \boxed{\mathbf{ab}} \\ \boxed{\mathbf{ba}}, & \boxed{\mathbf{bb}} \end{pmatrix}, \\
\boxed{\mathbf{aa}} &= \mathrm{Tr}[\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+], \\
\boxed{\mathbf{ab}} &= \mathrm{Tr}[\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{DWD}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+] = \boxed{\mathbf{ba}}, \\
\boxed{\mathbf{bb}} &= \mathrm{Tr}[\mathbf{DWD}'('\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{DWD}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+]
\end{aligned}$$

In the case of normality it is valid that

$$\mathrm{Var}_{\sigma_{\mathbf{0}}^{\mathbf{2}}}(\widehat{\sigma}_{\mathbf{2}}^{\mathbf{2}}) \leq \mathrm{Var}_{\sigma_{\mathbf{0}}^{\mathbf{2}}}(\widehat{\sigma}_{\mathbf{3}}^{\mathbf{2}})$$

The estimator based on $\mathbf{Y} - \mathbf{D}\widehat{\mathbf{\Theta}}$:

$$\widehat{\sigma}_4^2 = \frac{(\mathbf{Y} - \mathbf{D}\widehat{\mathbf{\Theta}})'(\mathbf{M}_X \mathbf{\Sigma}_0 \mathbf{M}_X)^+(\mathbf{Y} - \mathbf{D}\widehat{\mathbf{\Theta}}) - \mathrm{Tr}[(\mathbf{M}_X \mathbf{\Sigma}_0 \mathbf{M}_X)^+ \mathbf{DWD}']}{n - r(\mathbf{X})},$$

$$\begin{aligned} \mathrm{Var}_{\sigma_0^2}(\widehat{\sigma}_4^2) &= \frac{2\sigma^4}{n - r(\mathbf{X})} + \frac{4\sigma^2 \mathrm{Tr}[(\mathbf{M}_X \mathbf{\Sigma}_0 \mathbf{M}_X)^+ \mathbf{DWD}'(\mathbf{M}_x \mathbf{\Sigma}_0 \mathbf{M}_X)^+ \mathbf{V}]}{[n - r(\mathbf{X})]^2} \\ &+ \frac{2\mathrm{Tr}[(\mathbf{M}_X \mathbf{\Sigma}_0 \mathbf{M}_X)^+ \mathbf{DWD}'(\mathbf{M}_X \mathbf{\Sigma}_0 \mathbf{M}_X)^+ \mathbf{DWD}']}{[n - r(\mathbf{X})]^2} \end{aligned}$$

INSENSITIVITY REGION FOR THE DISPERSION OF THE ESTIMATOR OF LINEAR FUNCTIONS

Let $h(\boldsymbol{\beta}) = \mathbf{h}'\boldsymbol{\beta}, \quad \boldsymbol{\beta} \in R^k$. The neighbourhood $\mathcal{N}_{h'\beta}$ od the parameter σ_0^2 with the property

$$\sigma^2 \in \mathcal{N}_{h'\beta} \Rightarrow \sqrt{\mathrm{Var}_{\sigma_0^2}[\mathbf{h}'\widehat{\boldsymbol{\beta}}(\sigma^2)]} \leq (1+\varepsilon)\sqrt{\mathrm{Var}_{\sigma_0^2}[\mathbf{h}'\widehat{\boldsymbol{\beta}}(\sigma_0^2)]}$$

is called the insensitivity region.

In our case

$$\mathcal{N}_{h'\beta} = \Bigg\{\sigma^2 : |\sigma^2 - \sigma_0^2| \leq$$

$$\leq \sqrt{\frac{2\varepsilon\mathbf{h}'(\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{h}}{\mathbf{h}'(\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^{+}\mathbf{V}\boldsymbol{\Sigma}_0^{-1}\mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{h}}}\Bigg\}.$$

LINEARIZATION REGION FOR THE BIAS OF ESTIMATORS

Let instead the linear model

$$\mathbf{Y} \sim N\left[(\mathbf{D}, \mathbf{X})\begin{pmatrix} \boldsymbol{\Theta} \\ \boldsymbol{\beta} \end{pmatrix}, \boldsymbol{\Sigma}\right],$$

a nonlinear model

$$\mathbf{Y} \sim N(\mathbf{f}(\boldsymbol{\Theta}, \boldsymbol{\beta}), \boldsymbol{\Sigma})$$

be considered and let $\mathbf{f}(\cdot, \cdot\cdot)$ can be expressed as

$$\begin{aligned}
\mathbf{f}(\boldsymbol{\Theta}, \boldsymbol{\beta}) &= \mathbf{f}(\boldsymbol{\Theta}_0, \boldsymbol{\beta}_0) + \frac{\partial \mathbf{f}(\boldsymbol{\Theta}_0, \boldsymbol{\beta}_0)}{\partial \boldsymbol{\Theta}'}(\boldsymbol{\Theta} - \boldsymbol{\Theta}_0) + \frac{\partial \mathbf{f}(\boldsymbol{\Theta}_0, \boldsymbol{\beta}_0)}{\partial \boldsymbol{\beta}_0'}(\boldsymbol{\beta} - \boldsymbol{\beta}_0) \\
&\quad + \frac{1}{2}\begin{pmatrix} \vdots \\ \begin{pmatrix} \boldsymbol{\Theta} - \boldsymbol{\Theta}_0 \\ \boldsymbol{\beta} - \boldsymbol{\beta}_0 \end{pmatrix}' \frac{\partial^2 f_i(\boldsymbol{\Theta}_0, \boldsymbol{\beta}_0)}{\partial \begin{pmatrix} \boldsymbol{\Theta} \\ \boldsymbol{\beta} \end{pmatrix} \partial(\boldsymbol{\Theta}', \boldsymbol{\beta}')} \begin{pmatrix} \boldsymbol{\Theta} - \boldsymbol{\Theta}_0 \\ \boldsymbol{\beta} - \boldsymbol{\beta}_0 \end{pmatrix} \\ \vdots \end{pmatrix} \\
&= \mathbf{f}_0 + \mathbf{D}\delta\boldsymbol{\Theta} + \mathbf{X}\delta\boldsymbol{\beta} + \frac{1}{2}\boldsymbol{\kappa}(\delta\boldsymbol{\Theta}, \delta\boldsymbol{\beta}),
\end{aligned}$$

$$\delta\boldsymbol{\Theta} = \boldsymbol{\Theta} - \boldsymbol{\Theta}_0, \quad \delta\boldsymbol{\beta} = \boldsymbol{\beta} - \boldsymbol{\beta}_0,$$

with a sufficiently high accuracy. Then

$$\begin{aligned}
\mathbf{b} &= E(\delta\widehat{\boldsymbol{\beta}}) - \delta\boldsymbol{\beta} \\
&= E(\widehat{\boldsymbol{\beta}}) - \boldsymbol{\beta} = \frac{1}{2}[\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{X}]^{-1}\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{DWD}')^{-1}\boldsymbol{\kappa}(\delta\boldsymbol{\Theta}, \delta\boldsymbol{\beta}).
\end{aligned}$$

Let

$$C_b(\boldsymbol{\Theta}_0, \boldsymbol{\beta}_0) = \sup\left\{\frac{\sqrt{\mathbf{b}'\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{X}\mathbf{b}}}{(\delta\boldsymbol{\Theta}', \delta\boldsymbol{\beta}')\mathbf{U}^{-1}\begin{pmatrix} \delta\boldsymbol{\Theta} \\ \delta\boldsymbol{\beta} \end{pmatrix}} : \begin{pmatrix} \delta\boldsymbol{\Theta} \\ \delta\boldsymbol{\beta} \end{pmatrix} \in R^{l+k}\right\},$$

and at the same time

$$\begin{aligned}
\mathbf{U} &= \begin{pmatrix} \boxed{\mathbf{aa}}, & \boxed{\mathbf{ab}} \\ \boxed{\mathbf{ba}}, & \boxed{\mathbf{bb}} \end{pmatrix}, \\
\boxed{\mathbf{aa}} &= \mathbf{W}, \\
\boxed{\mathbf{ab}} &= -\mathbf{W}\mathbf{D}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{X}[\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{X}]^{-1} = \boxed{\mathbf{ba}}', \\
\boxed{\mathbf{bb}} &= [\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{X}]^{-1}.
\end{aligned}$$

In this case the linearization region for the bias of the estimator of the parameter $\boldsymbol{\beta}$ is

$$\mathcal{L}_b = \left\{ \begin{pmatrix} \delta\boldsymbol{\Theta} \\ \delta\boldsymbol{\beta} \end{pmatrix} : (\delta\boldsymbol{\Theta}', \delta\boldsymbol{\beta}')\mathbf{U}^{-1} \begin{pmatrix} \delta\boldsymbol{\Theta} \\ \delta\boldsymbol{\beta} \end{pmatrix} \leq \frac{\varepsilon}{C_b(\boldsymbol{\Theta}_0, \boldsymbol{\beta}_0)} \right\}$$

and it is valid that

$$\begin{pmatrix} \delta\boldsymbol{\Theta} \\ \delta\boldsymbol{\beta} \end{pmatrix} \in \mathcal{L}_b \Rightarrow \sqrt{\mathbf{b}'\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{X}\mathbf{b}} \leq \varepsilon.$$

Let

$$C_{1,\sigma^2}^{(int)} = \sup \left\{ \frac{\sqrt{\boldsymbol{\kappa}'(\delta\boldsymbol{\Theta}, \delta\boldsymbol{\beta}) \left(\mathbf{M}_{(D,X)}\mathbf{V}\mathbf{M}_{(D,X)}\right)^{+} \boldsymbol{\kappa}(\delta\boldsymbol{\Theta}, \delta\boldsymbol{\beta})}}{\delta\boldsymbol{\Theta}', \delta\boldsymbol{\beta}')\mathbf{U}^{-1} \begin{pmatrix} \delta\boldsymbol{\Theta} \\ \delta\boldsymbol{\beta} \end{pmatrix}} : \begin{pmatrix} \delta\boldsymbol{\Theta} \\ \delta\boldsymbol{\beta} \end{pmatrix} \in R^{l+k} \right\}.$$

In the case of normality the linearization region for the bias of the estimator $\widehat{\sigma}_1^2$ is

$$\mathcal{L}_{1,\sigma^2} = \left\{ \begin{pmatrix} \delta\boldsymbol{\Theta} \\ \delta\boldsymbol{\beta} \end{pmatrix} : (\delta\boldsymbol{\Theta}', \delta\boldsymbol{\beta}')\mathbf{U}^{-1} \begin{pmatrix} \delta\boldsymbol{\Theta} \\ \delta\boldsymbol{\beta} \end{pmatrix} \leq \frac{\sqrt{8[n - r(\mathbf{D}, \mathbf{X})]}}{C_{1,\sigma^2}^{int}} \right\}$$

and it is valid that

$$\begin{pmatrix} \delta\boldsymbol{\Theta} \\ \delta\boldsymbol{\beta} \end{pmatrix} \in \mathcal{L}_{1,\sigma^2} \Rightarrow \left| \sqrt{E(\widehat{\sigma}_1^2)} - \sigma \right| \leq \varepsilon\sigma.$$

Let

$$C_{2,\sigma^2}^{(int)} = \sup \left\{ \frac{\sqrt{\boldsymbol{\kappa}'(\delta\boldsymbol{\Theta}, \delta\boldsymbol{\beta})(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^{+}\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^{+}\boldsymbol{\kappa}(\delta\boldsymbol{\Theta}, \delta\boldsymbol{\beta})}}{\delta\boldsymbol{\Theta}', \delta\boldsymbol{\beta}')\mathbf{U}^{-1}\begin{pmatrix} \delta\boldsymbol{\Theta} \\ \delta\boldsymbol{\beta} \end{pmatrix}} : \begin{pmatrix} \delta\boldsymbol{\Theta} \\ \delta\boldsymbol{\beta} \end{pmatrix} \in R^{l+k} \right\}.$$

The linearization region for the bias of the estimator $\widehat{\sigma}_2^2$ is

$$\mathcal{L}_{2,\sigma^2} = \left\{ \begin{pmatrix} \delta\boldsymbol{\Theta} \\ \delta\boldsymbol{\beta} \end{pmatrix} : \right.$$

$$\left. (\delta\boldsymbol{\Theta}', \delta\boldsymbol{\beta}')\mathbf{U}^{-1}\begin{pmatrix} \delta\boldsymbol{\Theta} \\ \delta\boldsymbol{\beta} \end{pmatrix} \leq \frac{\sigma}{C_{2,\sigma^2}^{(int)}}\sqrt{2\varepsilon \mathrm{Tr}[\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^{+}\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^{+}]} \right\}$$

$$\begin{pmatrix} \delta\boldsymbol{\Theta} \\ \delta\boldsymbol{\beta} \end{pmatrix} \in \mathcal{L}_{2,\sigma^2} \Rightarrow \left| \sqrt{E(\widehat{\sigma}_2^2)} - \sigma \right| \leq \varepsilon\sigma.$$

NUMERICAL EXAMPLE

Let, in the plane, four points $A_1,\ A_2,\ A_3,\ A_4$ be given by their coordinates, i.e. $A_i\begin{pmatrix}\Theta_{2i-1}\\ \Theta_{2i}\end{pmatrix},\ i=1,2,3,4,$

$$A_1\begin{pmatrix}201.31m\\ 210.80m\end{pmatrix},\ A_2\begin{pmatrix}406.73m\\ 863.45m\end{pmatrix},\ A_3\begin{pmatrix}1050.47m\\ 216.66m\end{pmatrix},\ A_4\begin{pmatrix}630.17m\\ 28.29m\end{pmatrix}.$$

The coordinates are estimated and their estimator is

$$\widehat{\boldsymbol{\Theta}}=\begin{pmatrix}\widehat{\Theta}_1\\ \vdots\\ \widehat{\Theta}_8\end{pmatrix}\sim N_8(\boldsymbol{\Theta},\mathbf{W}),\qquad \mathbf{W}=(0.1m)^2\mathbf{I}_8.$$

Coordinates $\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}$ **of a point P must be estimated by measured distances d_i**

$$d_i=E(Y_i)=\sqrt{(\Theta_{2i-1}-\beta_1)^2+(\Theta_{2i}-\beta_2)^2},\ i=1,\dots,4,$$

where the approximate coordinates are

$$P\begin{pmatrix}503.1m\\ 431.9m\end{pmatrix},$$

$$\mathbf{Y}\sim N_4\left[\begin{pmatrix}d_1\\ \vdots\\ d_4\end{pmatrix},(0.01m)^2\mathbf{I}_4\right].$$

The linearized models of this measurement are

$$\mathbf{Y}-\mathbf{f}_0\sim N_4\left[(\mathbf{D},\mathbf{X})\begin{pmatrix}\delta\boldsymbol{\Theta}\\ \delta\boldsymbol{\beta}\end{pmatrix},(0.01m)^2\mathbf{I}_4\right]$$

and

$$\mathbf{Y}-\mathbf{f}_0-\mathbf{D}\widehat{\delta\boldsymbol{\Theta}}\sim N_4[\mathbf{X}\delta\boldsymbol{\beta},(0.01m)^2\mathbf{I}_4+\mathbf{D}(0.1m)^2\mathbf{I}_8\mathbf{D}'],$$

respectively.

Here

$$\mathbf{f}_0 = (f_{0,1}, \ldots, f_{0,4})' = (374.12, 442.18, 588.17, 423.14)',$$
$$f_{0,i} = \sqrt{\left(\Theta_{2i-1}^{(0)} - \beta_1^{(0)}\right)^2 + \left(\Theta_{2i}^{(0)} - \beta_2^{(0)}\right)^2},$$
$$\mathbf{D} = \frac{\partial \mathbf{f}(\boldsymbol{\Theta}, \boldsymbol{\beta})}{\partial \boldsymbol{\Theta}'}\Big|_{\Theta=\Theta^{(0)}, \beta=\beta^{(0)}},$$
$$\frac{\partial f_{0,i}}{\partial \Theta_{2i-1}} = \frac{\Theta_{2i-1}^{(0)} - \beta_1^{(0)}}{f_{0,i}}, \quad \frac{\partial f_{0,i}}{\partial \Theta_{2i}} = \frac{\Theta_{2i}^{(0)} - \beta_2^{(0)}}{f_{0,i}},$$
$$\frac{\partial f_{0,i}}{\partial \beta_1} = -\frac{\Theta_{2i-1}^{(0)} - \beta_1^{(0)}}{f_{0,i}}, \quad \frac{\partial f_{0,i}}{\partial \beta_2} = -\frac{\Theta_{2i}^{(0)} - \beta_2^{(0)}}{f_{0,i}},$$
$$\mathbf{D} = \begin{pmatrix} \frac{\partial f_{0,1}}{\partial \Theta_1}, & \frac{\partial f_{0,1}}{\partial \Theta_2}, & 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & \frac{\partial f_{0,2}}{\partial \Theta_3}, & \frac{\partial f_{0,2}}{\partial \Theta_4}, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & \frac{\partial f_{0,3}}{\partial \Theta_5}, & \frac{\partial f_{0,3}}{\partial \Theta_6}, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0, & \frac{\partial f_{0,4}}{\partial \Theta_7}, & \frac{\partial f_{0,4}}{\partial \Theta_8} \end{pmatrix},$$
$$\mathbf{DD}' = \mathbf{I}_4,$$
$$\mathbf{X} = \begin{pmatrix} \frac{\partial f_{0,1}}{\partial \beta_1}, & \frac{\partial f_{0,1}}{\partial \beta_2} \\ \frac{\partial f_{0,2}}{\partial \beta_1}, & \frac{\partial f_{0,2}}{\partial \beta_2} \\ \frac{\partial f_{0,3}}{\partial \beta_1}, & \frac{\partial f_{0,3}}{\partial \beta_2} \\ \frac{\partial f_{0,4}}{\partial \beta_1}, & \frac{\partial f_{0,4}}{\partial \beta_2} \end{pmatrix} = \begin{pmatrix} 0.806\ 68, & 0.59o\ 99 \\ 0.217\ 94, & -0.975\ 96 \\ -0.930\ 63, & 0.365\ 95 \\ -0.300\ 30, & 0.953\ 84 \end{pmatrix},$$
$$\boldsymbol{\Sigma}_0 = (0.01m)^2 \mathbf{I}_4 + (0.1m)^2 \mathbf{DD}' = 0.0101 \mathbf{I}_4.$$

The estimator $\widehat{\sigma_1^2}$ does not exist, since $\mathbf{M}_{(D,X)} = \mathbf{0}$.

The esimator $\widehat{\sigma_2^2}$ is

$$\widehat{\sigma_2^2} = \frac{A - B}{\mathrm{Tr}[(\mathbf{M}_X \boldsymbol{\Sigma}_0 \mathbf{M}_X)^+ \mathbf{V} (\mathbf{M}_X \boldsymbol{\Sigma}_0 \mathbf{M}_X)^+ \mathbf{V}]},$$
$$A = (\mathbf{Y} - \mathbf{D}\widehat{\boldsymbol{\Theta}})'(\mathbf{M}_X \boldsymbol{\Sigma}_0 \mathbf{M}_X)^+ \mathbf{V} (\mathbf{M}_X \boldsymbol{\Sigma}_0 \mathbf{M}_X)^+ (\mathbf{Y} - \mathbf{D}\widehat{\boldsymbol{\Theta}}),$$
$$B = \mathrm{Tr}[(\mathbf{M}_X \boldsymbol{\Sigma}_0 \mathbf{M}_X)^+ \mathbf{V} (\mathbf{M}_X \boldsymbol{\Sigma}_0 \mathbf{M}_X)^+ \mathbf{DWD}'],$$

and its dispersion is

$$\mathrm{Var}_{\sigma_0^2}(\widehat{\sigma_2^2}) = \frac{2\sigma_0^4}{\mathrm{Tr}[(\mathbf{M}_X \boldsymbol{\Sigma}_0 \mathbf{M}_X)^+ \mathbf{V} (\mathbf{M}_X \boldsymbol{\Sigma}_0 \mathbf{M}_X)^+ \mathbf{V}]} = 1.020\ 01 \times 10^{-4}.$$

The estimator $\widehat{\sigma_3^2}$ is

$$\widehat{\sigma_3^2} = (\mathbf{Y} - \mathbf{D}\widehat{\boldsymbol{\Theta}})'(\mathbf{M}_X \boldsymbol{\Sigma}_0 \mathbf{M}_X)^+ (\lambda_1 \mathbf{V} + \lambda_2 \mathbf{DWD}')(\mathbf{M}_X \boldsymbol{\Sigma}_0 \mathbf{M}_X)^+ (\mathbf{Y} - \mathbf{D}\widehat{\boldsymbol{\Theta}}),$$

$$\mathrm{Var}_{\sigma_0^2}(\widehat{\sigma_3^2}) = 2((1,0)\mathbf{S}_{V,DWD'}^{-1}\begin{pmatrix}1\\0\end{pmatrix} = 1.030\ 05\times 10^{-4},$$

where

$$\begin{aligned}
\mathbf{S}_{V,DWD'} &= \begin{pmatrix} \boxed{\mathbf{aa}}, & \boxed{\mathbf{ab}} \\ \boxed{\mathbf{ba}}, & \boxed{\mathbf{bb}} \end{pmatrix},\\
\boxed{\mathbf{aa}} &= \mathrm{Tr}[\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^{+}\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^{+}],\\
\boxed{\mathbf{ab}} &= \mathrm{Tr}[\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^{+}\mathbf{DWD}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^{+}] = \boxed{\mathbf{ba}},\\
\boxed{\mathbf{bb}} &= \mathrm{Tr}[\mathbf{DWD}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^{+}\mathbf{DWD}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^{+}].
\end{aligned}$$

The estimator $\widehat{\sigma_4^2}$ is

$$\widehat{\sigma_4^2} = \frac{(\mathbf{Y}-\mathbf{D}\widehat{\boldsymbol{\Theta}})'(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^{+}(\mathbf{Y}-\mathbf{D}\widehat{\boldsymbol{\Theta}}) - \mathrm{Tr}[(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^{+}\mathbf{DWD}']}{n-r(\mathbf{X})},$$

$$\begin{aligned}
\mathrm{Var}(\sigma_4^2) &= \frac{2\sigma^4}{n-r(\mathbf{X})} + \frac{4\sigma^2\mathrm{Tr}[(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^{+}\mathbf{DWD}'(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^{+}\mathbf{V}]}{[n-r(\mathbf{X})]^2}\\
&+\frac{2\mathrm{Tr}[\mathbf{DWD}'(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^{+}\mathbf{DWD}'(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^{+}]}{[n-r(\mathbf{X})]^2} = 1.020\ 01\times 10^{-4}.
\end{aligned}$$

The accuracy of different estimators of σ^2 is almost the same, even it is not sufficiently good. For example in the case of $\widehat{\sigma}^2$ it is approximately valid that

$$\frac{\sqrt{\mathrm{Var}(\sqrt{\widehat{\sigma}^2})}}{\sigma} = \sqrt{\left(\frac{1}{2\sqrt{\sigma^2}}\right)^2 \mathrm{Var}(\widehat{\sigma}_2^2)} = \frac{1}{2\times 10^{-2}}\sqrt{1.020\ 01\times 10^{-4}} = 0.505.$$

Thus the relative standard deviation is 50.5% what is rather large number. However nothing better can be expected because of the poor precision of the first stage measurement (W).

Insensitivity regions **Let**

$$W_h = \mathbf{h}'(\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0^{-1}\mathbf{M}_X)^{+}\mathbf{V}\boldsymbol{\Sigma}_0^{-1}\mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{h}.$$

If $\mathbf{h}=(1,0)'$, then $\mathbf{W}_{(1,0)}=\mathbf{0}$ and also for $\mathbf{h}=(0,1)'$ $\mathbf{W}_{(0,1)}=\mathbf{0}$.

Thus the estimators of β_1 and β_2 are not sensitive on the small changes of the value σ^2.

Linearization regions

$$\mathbf{U} = \begin{pmatrix} \boxed{\mathbf{aa}}, & \boxed{\mathbf{ab}} \\ \boxed{\mathbf{ba}}, & \boxed{\mathbf{bb}} \end{pmatrix},$$

$$\boxed{\mathbf{aa}} = \mathbf{W} = (0.1)^2\mathbf{I}_8,$$

$$\boxed{\mathbf{ab}} = -\mathbf{W}\mathbf{D}'(\boldsymbol{\Sigma}+\mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{X}[\mathbf{X}'(\boldsymbol{\Sigma}+\mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{X}]^{-1}$$

$$= \begin{pmatrix} 0.004\ 533, & 0.002\ 734 \\ 0.003\ 321, & 0.002\ 003 \\ 0.000\ 091, & -0.000\ 893 \\ -0.000\ 409, & 0.003\ 998 \\ 0.005\ 089, & -0.000\ 664 \\ -0.002\ 001, & 0.000\ 261 \\ 0.000\ 287, & -0.001\ 177 \\ -0.000\ 911, & 0.003\ 738 \end{pmatrix},$$

$$\boxed{\mathbf{ba}} = -[\mathbf{X}'(\boldsymbol{\Sigma}+\mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{X}]^{-1}\mathbf{X}'(\boldsymbol{\Sigma}+\mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{D}\mathbf{W} = \boxed{\mathbf{ab}}',$$

$$\boxed{\mathbf{bb}} = [\mathbf{X}'(\boldsymbol{\Sigma}+\mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{X}]^{-1} = \begin{pmatrix} 0.006\ 319, & 0.000\ 978 \\ 0.000\ 978, & 0.004\ 457 \end{pmatrix},$$

$$C_b(\boldsymbol{\Theta}^{(0)}, \boldsymbol{\beta}^{(0)}) = \sup\left\{ \frac{\sqrt{\mathbf{b}'\mathbf{X}'(\boldsymbol{\Sigma}+\mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{X}\mathbf{b}}}{(\delta\boldsymbol{\Theta}', \delta\boldsymbol{\beta}')\mathbf{U}^{-1}\begin{pmatrix}\delta\boldsymbol{\Theta}\\ \delta\boldsymbol{\beta}\end{pmatrix}} : \begin{pmatrix}\delta\boldsymbol{\Theta}\\ \delta\boldsymbol{\beta}\end{pmatrix} \in R^{l+k} \right\}$$

$$= 0.000\ 119\ 76.$$

If $\varepsilon = 0.1$, **then**

$$\mathcal{L}_b = \left\{ \begin{pmatrix}\delta\boldsymbol{\Theta}\\ \delta\boldsymbol{\beta}\end{pmatrix} : (\delta\boldsymbol{\Theta}', \delta\boldsymbol{\beta}')\mathbf{U}\begin{pmatrix}\delta\boldsymbol{\Theta}\\ \delta\boldsymbol{\beta}\end{pmatrix} \leq \frac{\varepsilon}{C_b(\boldsymbol{\Theta}^{(0)}, \boldsymbol{\beta}^{(0)})} \right\}$$

is the ellipsoid with the semiaxes equal to

$a_1 = 0.268m$, $a_2 = 0.292m$, $a_3 = 0.346m$, $a_4 = 0.346m$, $a_5 = 0.346m$,
$a_6 = 0.346m$, $a_7 = 0.346m$, $a_8 = 0.346m$, $a_9 = 5.487m$, $a_{10} = 6.530m$.

$$C_{2,\sigma^2} = \sup\left\{ \frac{\sqrt{\boldsymbol{\kappa}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\boldsymbol{\kappa}}}{(\delta\boldsymbol{\Theta}', \delta\boldsymbol{\beta}')\mathbf{U}^{-1}\begin{pmatrix}\delta\boldsymbol{\Theta}\\ \delta\boldsymbol{\beta}\end{pmatrix}} : \begin{pmatrix}\delta\boldsymbol{\Theta}\\ \delta\boldsymbol{\beta}\end{pmatrix} \in R^{l+k} \right\}$$

$$= \quad -0.003\ 776.$$

If $\varepsilon = 0.1$, **then**

$$\mathcal{L}_{2,\sigma^2} = \left\{ \begin{pmatrix} \delta\boldsymbol{\Theta} \\ \delta\boldsymbol{\beta} \end{pmatrix} : (\delta\boldsymbol{\Theta}', \delta\boldsymbol{\beta}')\mathbf{U}^{-1} \begin{pmatrix} \delta\boldsymbol{\Theta} \\ \delta\boldsymbol{\beta} \end{pmatrix} \right.$$
$$\left. \leq \frac{\sigma}{C_{2,\sigma^2}} \sqrt{2\varepsilon \mathrm{Tr}[\mathbf{V}(\mathbf{M}_X \boldsymbol{\Sigma}_0 \mathbf{M}_X)^+ \mathbf{V}(\mathbf{M}_X \boldsymbol{\Sigma}_0 \mathbf{M}_X)^+]} \right\}$$

is the ellipsoid with the semiaxes

$a_1 = 1.504m, \quad a_2 = 1.642, \quad a_3 = 1.943m \quad a_4 = 1.943m, \quad a_5 = 1.943m,$
$a_6 = 1.943m, \quad a_7 = 1.943m, \quad a_8 = 1.943m, \quad a_9 = 30.813m, \quad a_{10} = 36.668m.$

The linearization regions $\mathcal{L}_b$ **and** $\mathcal{L}_{2,\sigma^2}$ **are sufficiently large with respect to requirements of geodetical practice.**

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