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Department of Mathematical analysis
and Applications of Mathematics
Faculty of Science
Palacký University Olomouc

Copulas and their applications in multicriteria decision support

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ODAM 2011, Olomouc, Czech Republic

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Introduction

Abe Sklar 1957

copula $C : [0, 1]^n \to [0, 1]$

Sklar theorem

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Axiomatic approach

i) $C(x_1, \ldots, x_n) = 0$ if some $x_i = 0$
C is **grounded**

ii) $C(x_1, \ldots, x_n) = x_i$ if $\forall j \neq i, x_j = 1$
C has **neutral element 1**

iii) $\forall u_1 \leq v_1, \ldots, u_n \leq v_n$
$\sum (-1)^{|I|} C(t_1^I, \ldots, t_n^I) \geq 0$
$I \subseteq \{1, \ldots, n\} \quad t_i^I = \begin{cases} u_i & \text{if } i \in I, \\ v_i & \text{if } i \notin I \end{cases}$
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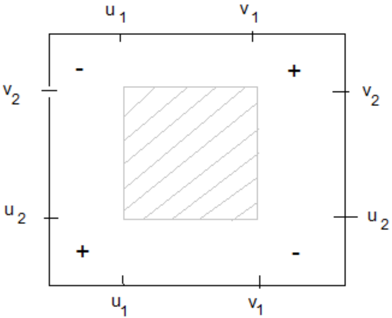


Each C is **1–Lipschitz** (wrt. L_1–norm)

$$C(v_1, v_2) - C(v_1, u_2) - C(v_2, u_1) + C(u_1, u_2) \geq 0$$

2–increasing, supermodular

$$W \leq C \leq M$$

$$M(u_1, \ldots, u_n) = min\ \{u_1, \ldots, u_n\}$$

$$W(u_1, \ldots, u_n) = max\ \left\{0, \sum_{i=1}^{n} u_i - n + 1\right\}$$

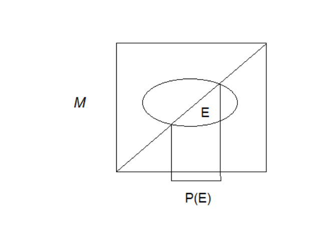
$$\Pi(u_1, \ldots, u_n) = \prod_{i=1}^{n} u_i$$

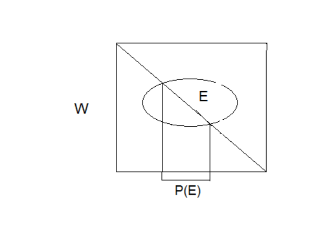
$\mathcal{C}_n$ class of all n–ary copulas is convex and compact

Example

Copulas are in 1–1 correspondence with probability measures on $\mathcal{B}([0,1]^n)$ with uniform 1–dimensional marginals

$$\Pi \sim \text{Lebesgue}$$





Π absolutely continuous, M, W singular

Statistical interpretation

$\Pi \sim$ independence

$M \sim$ comonotone dependence,

$$X_i = f_i(X_1), \quad f_i \nearrow$$

$W \sim$ counter monotone dependence,

$$X_2 = g(X_1), \quad g \searrow$$

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Gaussian and t–copulas $\sim$ eliptic copulas, no closed form

$$C(u_1,\ldots,u_n) = F_Z\left(F_1^{-1}(u_1),\ldots,F_n^{-1}(u_n)\right)$$

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flipping

$$C^{-}(u,v) = u - C(u, 1-v)$$

$$(X,Y) \to (X,-Y)$$

$$C_{-}(u,v) = v - C(1-u, v)$$

$$(X,Y) \to (-X,Y)$$

survival

$$\hat{C}(u,v) = u + v - 1 + C(1-u, 1-v)$$

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Ordinal sums

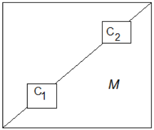

Figure: M–ordinal

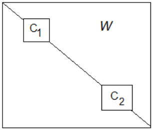

Figure: W–ordinal

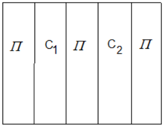

Figure: Π–ordinal

generators $f : [0, 1] \longrightarrow [0, \infty]$
strictly decreasing, continuous,

$$f(1) = 0$$

$$C_f(u_1, \ldots, u_n) = f^{-1}\left(min\left\{f(0), \sum_{i=1}^{n} f(u_i)\right\}\right)$$

$$g : [-\infty, 0] \to [0, 1], \quad , g(x) = f^{-1}(min\ \{f(0), -x\})$$

$$g' \geq 0, \ldots, g^{(n)} \geq 0$$

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Archimedean copulas

associative and $C_f(u,\dots,u) < u$ for $u \in]0,1[$

$$f_\Pi(x) = -log\,x \quad \forall n$$
$$f_W(x) = 1 - x \quad \text{only } n = 2$$

n fixed, weakest $C_{f^{[n]}}$

Clayton copula

$$f^{[n]}(x) = 1 - x^{\frac{1}{n-1}}$$

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$$\lambda \in [1, \infty[\quad f_\lambda^G(x) = (-log\, x)^\lambda, \quad \forall n$$

Clayton

$$f_\lambda^C(x) = \frac{x^{-\lambda} - 1}{\lambda}, \quad \lambda \in [-1, \infty[, \quad \lambda \neq 0$$

$$f_0^\lambda = f_\Pi, \quad \lambda \geq 0 \forall n$$

$\lambda = 1$ ALI–MIKHAIL–HAQ copula

Hamacher product

$$C(u, v) = \frac{uv}{u + v - uv}$$

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$$C_{(f)}(u,v) = u\,f^{-1}\left(min\left\{\frac{f(v)}{u}, f(0)\right\}\right)$$

Univariate Conditioning Stable

$$C_{(f_1^C)}(u,v) = \frac{u^2 v}{1-u+uv}$$

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Extreme Values copulas
Archimedean copulas
DUCS copulas

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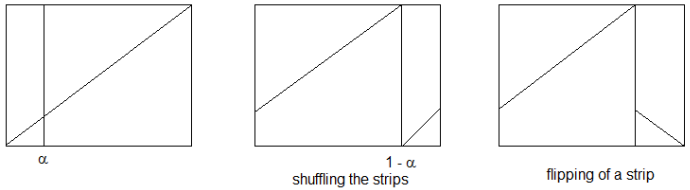
Special singular copulas

Shuffles of M



Each copula can be seen as a limit of shuffles of M

In multicriteria decision making with n criteria, $N = \{1, \ldots, n\}$, normalized compatible scales $[0, 1]$, $m : 2^N \to [0, 1]$

"boolean utility"

$$m(\emptyset) = 0, \; m(N) = 1,$$

$$E \subset F \Rightarrow m(E) \leq M(F)$$

m capacity, fuzzy measure, weights of criteria groups, monotone game ...

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aim: extend m to graded utility function

$$I_m : [0,1]^n \longrightarrow [0,1]$$

unanymous (idempotent)
nondecreasing (Pareto)
$I_m(c\,1_E)$ depends on c and $m(E)$ only!
then INTEGRAL
Klement, Mesiar, Pap ≈ Universal integrals ≈2010 in IEEE TFS

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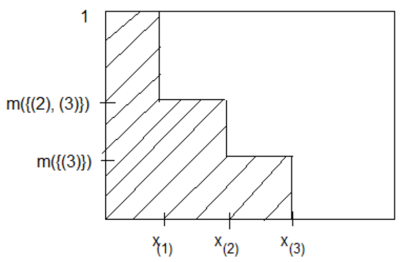
$$C : [0,1]^2 \to [0,1] \text{ a fixed copula}$$

$$I_{C,m}(\mathbf{x}) = P_C\left(\{(x,y) \in [0,1]^2 | y \le m(\{i \in N | x_i \ge x\})\}\right)$$



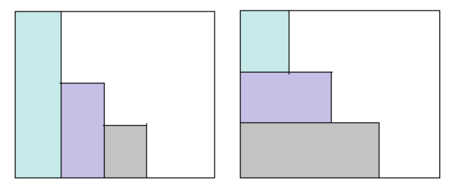
$(\cdot) : N \to N$ a permutation

$$x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$$

$$I_{C,m}(\mathbf{x}) = \sum_{i=1}^{n} \left(C(x_{(i)}, m(\{(i), \dots, (n)\}) - C(x_{(i-1)}, m(\{(i), \dots, (n)\}) \right) =$$

$$= \sum_{i=1}^{n} \left(C(x_{(i)}, m(\{(i), \dots, (n)\}) - C(x_{(i)}, m(\{(i+1), \dots, (n)\}) \right)$$

$$x_{(0)} \equiv 0, \quad m(\{(n+1), (n)\}) \equiv 0$$

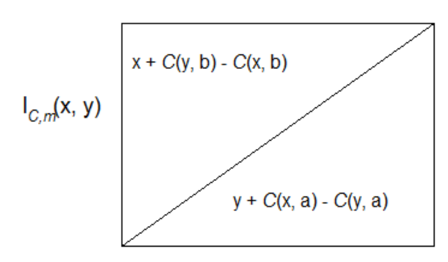


$I_{\Pi,m}$ CHOQUET integral
$I_{M,m}$ SUGENO integral

$$I_{C,m}(t \cdot 1_E) = C(t, m(E))$$

C expresses the dependence between function values and measure values

$$n = 2, \quad m(\{1\}) = a, \quad m(\{2\}) = b, \; a, b \in [0, 1]$$



m symmetric
= depends only on number of considered criteria,

$$|E| = |F| \Rightarrow m(E) = m(F)$$

$I_{C,m}$ anonymous (symmetric)
$I_{C,m} \equiv OMA$
Ordered Modular Average
comonotone modular, symmetric and idempotent aggregation function
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$(w_1, \ldots, w_n)$ discrete probability vector,

$$m(E) = \sum_{i=1}^{n} w_{n-i+1} = v_{n-|E|+1}$$

$$I_{\Pi,m}(\mathbf{x}) = \sum_{i=1}^{n} x_{(i)} w_i$$

OWA Yager 1988

$$I_{M,m}(\mathbf{x}) = \bigvee_{i=1}^{n} (x_{(i)} \wedge v_i)$$

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In general, for a symmetric m,

$$I_{C,m}(\mathbf{x}) = \sum_{i=1}^{n} \left(C(x_{(i)}, v_i) - C(x_{(i)}, v_{i+1})\right) = \sum_{i=1}^{n} f_i(x_{(i)}) = \sum_{i=1}^{n} w_i \cdot g_i(x_{(i)})$$

$$f_i : [0,1] \to [0,1], \text{ nondecreasing, } \textstyle\sum_{i=1}^{n} f_i = id$$
$$g_i : [0,1] \to [0,1], \text{ nondecreasing surjection}$$

$$w_i \cdot g_i = f_i$$

convention $v_{n+1} = 0$

$$n = 2 \quad (w_1, w_2)$$

$$I_{C,m}(x_1, x_2) = C(x_{(1)}, 1) - C(x_{(1)}, w_2) + C(x_{(2)}, w_2) - C(x_{(2)}, 0) =$$

$$= \left(x_{(1)} - C(x_{(1)}, w_2)\right) + C(x_{(2)}, w_2)$$

$$I_{W,m}(x_1, x_2) = f_1(x_{(1)}) + f_2(x_{(2)})$$

$$f_1(x) = min\ \{x, w_1\}$$

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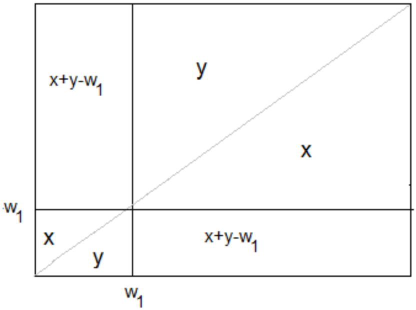
$I_{W,m}(x, y)$



Concluding remarks

We have

- discussed copulas
- introduced copula–based integrals generalizing Choquet and Sugeno integrals
- there is also axiomatic approach to the introduced integrals = OMA operators!

Thanks for your attention