european social fund in the czech republic

EUROPEAN UNION

MINISTRY OF EDUCATION, YOUTH AND SPORTS

OP Education for Competitiveness

INVESTMENTS IN EDUCATION DEVELOPMENT

Streamlining the Applied Mathematics Studies at Faculty of Science of Palacký University in Olomouc
CZ.1.07/2.2.00/15.0243

International Conference Olomoucian Days of Applied Mathematics

ODAM 2011

Department of Mathematical analysis and Applications of Mathematics
Faculty of Science
Palacký University Olomouc

Higher-Order Fuzzy Logic

Vilém Novák

University of Ostrava in Ostrava
Institute for Research and Applications of Fuzzy Modeling
30. dubna 22, 701 03 Ostrava 1, Czech Republic
Vilem.Novak@osu.cz

ODAM, Olomouc, January 26-28, 2011

Outline

What is fuzzy logic

Fuzzy logic is

a special many-valued logic whose aim is to provide means that can be used for modeling of various aspects of the vagueness phenomenon via the use of degrees of truth

FL in narrow sense — FLn

(a) *propositional:* traditional or evaluated syntax

(b) *predicate:* traditional or evaluated syntax

(c) *higher order* (fuzzy type theory)

Generalization of classical Simple Type Theory

What is fuzzy logic

Fuzzy logic is

a special many-valued logic whose aim is to provide means that can be used for modeling of various aspects of the vagueness phenomenon via the use of degrees of truth

FL in narrow sense — FLn

(a) *propositional:* traditional or evaluated syntax

(b) *predicate:* traditional or evaluated syntax

(c) *higher order* (fuzzy type theory)

Generalization of classical Simple Type Theory

History

- B. Russel (1903,1908)
- A. Church (1940), L. Henkin (1950, 1963), P. Andrews, P. Martin-Löf, W. Farmer

(i) Type theory as a (higher-order) **logic**

(ii) Type theory as effective theoretical tool in **computer science**

L. Henkin (1963)

Type theory with equality as its **sole** connective

History

- B. Russel (1903,1908)
- A. Church (1940), L. Henkin (1950, 1963), P. Andrews, P. Martin-Löf, W. Farmer

(i) Type theory as a (higher-order) **logic**

(ii) Type theory as effective theoretical tool in **computer science**

L. Henkin (1963)

Type theory with equality as its **sole** connective

History

- B. Russel (1903,1908)
- A. Church (1940), L. Henkin (1950, 1963), P. Andrews, P. Martin-Löf, W. Farmer

(i) Type theory as a (higher-order) **logic**

(ii) Type theory as effective theoretical tool in **computer science**

L. Henkin (1963)

Type theory with equality as its **sole** connective

History

- B. Russel (1903,1908)
- A. Church (1940), L. Henkin (1950, 1963), P. Andrews, P. Martin-Löf, W. Farmer

(i) Type theory as a (higher-order) **logic**

(ii) Type theory as effective theoretical tool in **computer science**

L. Henkin (1963)

Type theory with equality as its **sole** connective

*Why **FUZZY** type theory*

1. FTT provides a more transparent model of some deep manifestations of the vagueness phenomenon (including higher order vagueness)
2. *Logical analysis of concepts and natural language expressions requires higher-order logic (TT).* Replacing TT by FTT makes enables us to include vagueness in the developed models.
3. FLb (Fuzzy Logic in Broader Sense) develops a model of natural language semantics using FTT. We may thus bring a formal theory of commonsense reasoning closer to the human way of thinking.
4. Foundations of the whole "fuzzy" mathematics require higher order fuzzy logic. The expressive power of FTT makes the task easier.

*Why **FUZZY** type theory*

1. FTT provides a more transparent model of some deep manifestations of the vagueness phenomenon (including higher order vagueness)
2. *Logical analysis of concepts and natural language expressions requires higher-order logic (TT).* Replacing TT by FTT makes enables us to include vagueness in the developed models.
3. FLb (Fuzzy Logic in Broader Sense) develops a model of natural language semantics using FTT. We may thus bring a formal theory of commonsense reasoning closer to the human way of thinking.
4. Foundations of the whole "fuzzy" mathematics require higher order fuzzy logic. The expressive power of FTT makes the task easier.

*Why **FUZZY** type theory*

1. FTT provides a more transparent model of some deep manifestations of the vagueness phenomenon (including higher order vagueness)
2. *Logical analysis of concepts and natural language expressions requires higher-order logic (TT).* Replacing TT by FTT makes enables us to include vagueness in the developed models.
3. FLb (Fuzzy Logic in Broader Sense) develops a model of natural language semantics using FTT. We may thus bring a formal theory of commonsense reasoning closer to the human way of thinking.
4. Foundations of the whole "fuzzy" mathematics require higher order fuzzy logic. The expressive power of FTT makes the task easier.

*Why **FUZZY** type theory*

1. FTT provides a more transparent model of some deep manifestations of the vagueness phenomenon (including higher order vagueness)
2. *Logical analysis of concepts and natural language expressions requires higher-order logic (TT).* Replacing TT by FTT makes enables us to include vagueness in the developed models.
3. FLb (Fuzzy Logic in Broader Sense) develops a model of natural language semantics using FTT. We may thus bring a formal theory of commonsense reasoning closer to the human way of thinking.
4. Foundations of the whole “fuzzy” mathematics require higher order fuzzy logic. The expressive power of FTT makes the task easier.

Basic concepts

1. Truth degrees
2. Fuzzy equality
3. Types
4. Semantics based on frame (hierarchical structure of sets)
5. Syntax based on λ-calculus

Basic concepts

1. Truth degrees
2. Fuzzy equality
3. Types
4. Semantics based on frame (hierarchical structure of sets)
5. Syntax based on λ-calculus

Basic concepts

1. Truth degrees
2. Fuzzy equality
3. Types
4. Semantics based on frame (hierarchical structure of sets)
5. Syntax based on λ-calculus

Basic concepts

1. Truth degrees
2. Fuzzy equality
3. Types
4. Semantics based on frame (hierarchical structure of sets)
5. Syntax based on λ-calculus

Basic concepts

1. Truth degrees
2. Fuzzy equality
3. Types
4. Semantics based on frame (hierarchical structure of sets)
5. Syntax based on λ-calculus

Structure of truth degrees

Residuated lattice

$$\mathcal{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, \mathbf{0}, \mathbf{1} \rangle$$

(Integral, commutative, bounded, residuated lattice)

1. $\mathcal{E} = \langle E, \vee, \wedge, \mathbf{0}, \mathbf{1} \rangle$ — lattice with $\mathbf{0}, \mathbf{1}$
2. $\otimes$ is associative, commutative, $a \otimes \mathbf{1} = a$
3. adjunction: $a \otimes b \leq c \quad$ iff $\quad a \leq b \rightarrow c$
   (algebraic formulation of modus ponens)

Essential kinds of algebras of truth degrees

(i) prelinearity $(a \to b) \vee (b \to a) = \mathbf{1}$
(ii) divisibility $a \otimes (a \to b) = a \wedge b$
(iii) double negation $\neg\neg a = a$

- *MTL-algebra* — prelinearity
- *IMTL-algebra* — prelinearity + double negation
- *BL-algebra* — prelinearity + divisibility
- *MV-algebra* — prelinearity + divisibility+double negation

Essential kinds of algebras of truth degrees

(i) prelinearity $(a \to b) \vee (b \to a) = \mathbf{1}$
(ii) divisibility $a \otimes (a \to b) = a \wedge b$
(iii) double negation $\neg\neg a = a$

- *MTL-algebra* — prelinearity
- *IMTL-algebra* — prelinearity + double negation
- *BL-algebra* — prelinearity + divisibility
- *MV-algebra* — prelinearity + divisibility+double negation

Essential kinds of algebras of truth degrees

(i) prelinearity $(a \rightarrow b) \vee (b \rightarrow a) = \mathbf{1}$
(ii) divisibility $a \otimes (a \rightarrow b) = a \wedge b$
(iii) double negation $\neg\neg a = a$

- *MTL-algebra* — prelinearity
- *IMTL-algebra* — prelinearity + double negation
- *BL-algebra* — prelinearity + divisibility
- *MV-algebra* — prelinearity + divisibility+double negation

Essential kinds of algebras of truth degrees

(i) prelinearity $(a \to b) \vee (b \to a) = \mathbf{1}$

(ii) divisibility $a \otimes (a \to b) = a \wedge b$

(iii) double negation $\neg\neg a = a$

- *MTL-algebra* — prelinearity
- *IMTL-algebra* — prelinearity + double negation
- *BL-algebra* — prelinearity + divisibility
- *MV-algebra* — prelinearity + divisibility+double negation

Essential kinds of algebras of truth degrees

(i) prelinearity $(a \to b) \vee (b \to a) = \mathbf{1}$
(ii) divisibility $a \otimes (a \to b) = a \wedge b$
(iii) double negation $\neg\neg a = a$

- *MTL-algebra* — prelinearity
- *IMTL-algebra* — prelinearity + double negation
- *BL-algebra* — prelinearity + divisibility
- *MV-algebra* — prelinearity + divisibility+double negation

EQ-algebras — special algebras for FTT

EQ-algebra

Algebra

$$\mathcal{E} = \langle E, \wedge, \otimes, \sim, \mathbf{1} \rangle$$

of type (2, 2, 2, 0)

Fuzzy equality is the main (basic) operation

EQ-algebras

Definition

(E1) $\langle E, \wedge \rangle$ is a $\wedge$-semilattice with the top element $\mathbf{1}$

(E2) $\langle E, \otimes, \mathbf{1} \rangle$ is a monoid
$\otimes$ is isotone w.r.t. $\leq$ ($a \leq b$ iff $a \wedge b = a$)

(E3) $a \sim a = \mathbf{1}$ (reflexivity)

(E4) $((a \wedge b) \sim c) \otimes (d \sim a) \leq c \sim (d \wedge b)$ (substitution)

(*Leibniz rule of indiscernibility of identicals*)

(E5) $(a \sim b) \otimes (c \sim d) \leq (a \sim c) \sim (b \sim d)$ (congruence)

(E6) $(a \wedge b \wedge c) \sim a \leq (a \wedge b) \sim a$ (monotonicity)

(E7) $a \otimes b \leq a \sim b$ (boundedness)

EQ-algebras

Definition

(E1) $\langle E, \wedge \rangle$ is a $\wedge$-semilattice with the top element $\mathbf{1}$

(E2) $\langle E, \otimes, \mathbf{1} \rangle$ is a monoid
$\otimes$ is isotone w.r.t. $\leq$ ($a \leq b$ iff $a \wedge b = a$)

(E3) $a \sim a = \mathbf{1}$ (reflexivity)

(E4) $((a \wedge b) \sim c) \otimes (d \sim a) \leq c \sim (d \wedge b)$ (substitution)

(*Leibniz rule of indiscernibility of identicals*)

(E5) $(a \sim b) \otimes (c \sim d) \leq (a \sim c) \sim (b \sim d)$ (congruence)

(E6) $(a \wedge b \wedge c) \sim a \leq (a \wedge b) \sim a$ (monotonicity)

(E7) $a \otimes b \leq a \sim b$ (boundedness)

EQ-algebras

Definition

(E1) $\langle E, \wedge \rangle$ is a $\wedge$-semilattice with the top element $\mathbf{1}$

(E2) $\langle E, \otimes, \mathbf{1} \rangle$ is a monoid
$\otimes$ is isotone w.r.t. $\leq$ $\quad$ ($a \leq b$ iff $a \wedge b = a$)

(E3) $a \sim a = \mathbf{1}$ $\quad$ (reflexivity)

(E4) $((a \wedge b) \sim c) \otimes (d \sim a) \leq c \sim (d \wedge b)$ $\quad$ (substitution)

(*Leibniz rule of indiscernibility of identicals*)

(E5) $(a \sim b) \otimes (c \sim d) \leq (a \sim c) \sim (b \sim d)$ $\quad$ (congruence)

(E6) $(a \wedge b \wedge c) \sim a \leq (a \wedge b) \sim a$ $\quad$ (monotonicity)

(E7) $a \otimes b \leq a \sim b$ $\quad$ (boundedness)

IRAFM

Special definitions in EQ-algebras

- $\tilde{a} = a \sim \mathbf{1}$
- $a \rightarrow b = (a \wedge b) \sim a$ (implication)
- If $\mathcal{E}$ contains $\mathbf{0}$ then $\neg a = a \sim \mathbf{0}$ (negation)
- $a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$ (biimplication)
- $a \Leftrightarrow b = (a \rightarrow b) \otimes (b \rightarrow a)$ (weak biimplication)

Special definitions in EQ-algebras

- $\tilde{a} = a \sim \mathbf{1}$
- $a \to b = (a \wedge b) \sim a$ (implication)
- If $\mathcal{E}$ contains $\mathbf{0}$ then $\neg a = a \sim \mathbf{0}$ (negation)
- $a \leftrightarrow b = (a \to b) \wedge (b \to a)$ (biimplication)
- $a \Leftrightarrow b = (a \to b) \otimes (b \to a)$ (weak biimplication)

Special definitions in EQ-algebras

- $\tilde{a} = a \sim \mathbf{1}$
- $a \to b = (a \wedge b) \sim a$ (implication)
- If $\mathcal{E}$ contains $\mathbf{0}$ then $\neg a = a \sim \mathbf{0}$ (negation)
- $a \leftrightarrow b = (a \to b) \wedge (b \to a)$ (biimplication)
- $a \Leftrightarrow b = (a \to b) \otimes (b \to a)$ (weak biimplication)

Special definitions in EQ-algebras

- $\tilde{a} = a \sim \mathbf{1}$
- $a \to b = (a \wedge b) \sim a$ (implication)
- If $\mathcal{E}$ contains $\mathbf{0}$ then $\neg a = a \sim \mathbf{0}$ (negation)
- $a \leftrightarrow b = (a \to b) \wedge (b \to a)$ (biimplication)
- $a \Leftrightarrow b = (a \to b) \otimes (b \to a)$ (weak biimplication)

Basic properties

Theorem

(a) $a \sim b = b \sim a$ *(symmetry)*

(b) $(a \sim b) \otimes (b \sim c) \leq (a \sim c)$ *(transitivity)*

(c) $(a \to b) \otimes (b \to c) \leq a \to c$ *(transitivity of $\to$)*

(d) $a \to b \leq (a \wedge c) \to b$ *(antitonicity of $\to$)*

"Semi- adjunction"

If $a \leq b \to c$, then $a \otimes b \leq \tilde{c}$

Basic properties

Theorem

(a) $a \sim b = b \sim a$ *(symmetry)*

(b) $(a \sim b) \otimes (b \sim c) \leq (a \sim c)$ *(transitivity)*

(c) $(a \rightarrow b) \otimes (b \rightarrow c) \leq a \rightarrow c$ *(transitivity of $\rightarrow$)*

(d) $a \rightarrow b \leq (a \wedge c) \rightarrow b$ *(antitonicity of $\rightarrow$)*

"Semi- adjunction"

If $a \leq b \rightarrow c$, then $a \otimes b \leq \tilde{c}$

Special EQ-algebras

EQ-algebra is:

(i) separated if $a \sim b = \mathbf{1}$ iff $a = b$

(ii) good if $a \sim \mathbf{1} = a$

(iii) prelinear if for all $a, b \in E$

$$\sup\{a \to b, b \to a\} = \mathbf{1}$$

(iv) involutive if $\neg\neg a = a$ (**IEQ-algebra**)

(v) residuated if
$(a \otimes b) \wedge c = a \otimes b$ iff $a \wedge ((b \wedge c) \sim b) = a$

(vi) *complete* if it is $\wedge$-semilattice complete.

(vii) ℓEQ-algebra if it is lattice ordered and

(E8) $((a \vee b) \sim c) \otimes (d \sim a) \leq ((d \vee b) \sim c)$

Special EQ-algebras

EQ-algebra is:

(i) separated if $a \sim b = \mathbf{1}$ iff $a = b$

(ii) good if $a \sim \mathbf{1} = a$

(iii) prelinear if for all $a, b \in E$

$$\sup\{a \to b, b \to a\} = \mathbf{1}$$

(iv) involutive if $\neg\neg a = a$ (**IEQ-algebra**)

(v) residuated if
$(a \otimes b) \wedge c = a \otimes b$ iff $a \wedge ((b \wedge c) \sim b) = a$

(vi) *complete* if it is $\wedge$-semilattice complete.

(vii) ℓEQ-algebra if it is lattice ordered and
(E8) $((a \vee b) \sim c) \otimes (d \sim a) \leq ((d \vee b) \sim c)$

Special EQ-algebras

EQ-algebra is:

(i) separated if $a \sim b = \mathbf{1} \quad \text{iff} \quad a = b$

(ii) good if $a \sim \mathbf{1} = a$

(iii) prelinear if for all $a, b \in E$

$$\sup\{a \to b, b \to a\} = \mathbf{1}$$

(iv) involutive if $\neg\neg a = a$ (**IEQ-algebra**)

(v) residuated if

$(a \otimes b) \wedge c = a \otimes b \quad \text{iff} \quad a \wedge ((b \wedge c) \sim b) = a$

(vi) *complete* if it is $\wedge$-semilattice complete.

(vii) ℓEQ-algebra if it is lattice ordered and

(E8) $((a \vee b) \sim c) \otimes (d \sim a) \leq ((d \vee b) \sim c)$

Special EQ-algebras

EQ-algebra is:

(i) separated if $a \sim b = \mathbf{1}$ iff $a = b$

(ii) good if $a \sim \mathbf{1} = a$

(iii) prelinear if for all $a, b \in E$

$$\sup\{a \to b, b \to a\} = \mathbf{1}$$

(iv) involutive if $\neg\neg a = a$ **(IEQ-algebra)**

(v) residuated if
$(a \otimes b) \wedge c = a \otimes b$ iff $a \wedge ((b \wedge c) \sim b) = a$

(vi) *complete* if it is $\wedge$-semilattice complete.

(vii) ℓEQ-algebra if it is lattice ordered and

(E8) $((a \vee b) \sim c) \otimes (d \sim a) \leq ((d \vee b) \sim c)$

Special EQ-algebras

EQ-algebra is:

(i) separated if $a \sim b = \mathbf{1} \quad \text{iff} \quad a = b$

(ii) good if $a \sim \mathbf{1} = a$

(iii) prelinear if for all $a, b \in E$

$$\sup\{a \to b, b \to a\} = \mathbf{1}$$

(iv) involutive if $\neg\neg a = a$ (**IEQ-algebra**)

(v) residuated if
$(a \otimes b) \wedge c = a \otimes b \quad \text{iff} \quad a \wedge ((b \wedge c) \sim b) = a$

(vi) *complete* if it is $\wedge$-semilattice complete.

(vii) ℓEQ-algebra if it is lattice ordered and

(E8) $((a \vee b) \sim c) \otimes (d \sim a) \leq ((d \vee b) \sim c)$

Special EQ-algebras

EQ-algebra is:

- **(i)** separated if $a \sim b = \mathbf{1}$ iff $a = b$
- **(ii)** good if $a \sim \mathbf{1} = a$
- **(iii)** prelinear if for all $a, b \in E$

$$\sup\{a \to b, b \to a\} = \mathbf{1}$$

- **(iv)** involutive if $\neg\neg a = a$ (**IEQ-algebra**)
- **(v)** residuated if
  $(a \otimes b) \wedge c = a \otimes b$ iff $a \wedge ((b \wedge c) \sim b) = a$
- **(vi)** *complete* if it is $\wedge$-semilattice complete.
- **(vii)** ℓEQ-algebra if it is lattice ordered and
  (E8) $((a \vee b) \sim c) \otimes (d \sim a) \leq ((d \vee b) \sim c)$

Special EQ-algebras

EQ-algebra is:

- **(i)** separated if $a \sim b = \mathbf{1}$ iff $a = b$
- **(ii)** good if $a \sim \mathbf{1} = a$
- **(iii)** prelinear if for all $a, b \in E$

$$\sup\{a \to b, b \to a\} = \mathbf{1}$$

- **(iv)** involutive if $\neg\neg a = a$ (**IEQ-algebra**)
- **(v)** residuated if
  $(a \otimes b) \wedge c = a \otimes b$ iff $a \wedge ((b \wedge c) \sim b) = a$
- **(vi)** *complete* if it is $\wedge$-semilattice complete.
- **(vii)** ℓEQ-algebra if it is lattice ordered and
  (E8) $((a \vee b) \sim c) \otimes (d \sim a) \leq ((d \vee b) \sim c)$

Representation of prelinear EQ (EQ_Δ)-algebras

Theorem (M. El Zekey)

Let $\mathcal{E}$ be a good EQ-algebra. The following are equivalent:

(a) *$\mathcal{E}$ is subdirectly embeddable into a product of linearly ordered good EQ-algebras.*

(b) *$\mathcal{E}$ satisfies*

$$(a \to b) \vee (d \to (d \otimes (c \to ((b \to a) \otimes c)))) = \mathbf{1}$$

Examples of EQ-algebras

Each residuated lattice

$$\mathcal{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, \mathbf{0}, \mathbf{1} \rangle$$

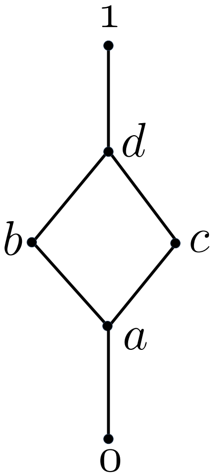
is a good residuated ℓ-EQ-algebra

$$\mathcal{E} = \langle E, \wedge, \otimes, \sim, \mathbf{1} \rangle$$

with fuzzy equality $\sim$ identified with the **biresiduation**

$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$$

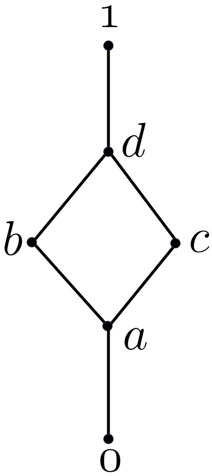
Examples of EQ-algebras — 6-element non-separated EQ-algebra



$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	a	a	a	b
c	0	0	a	0	a	c
d	0	0	a	a	a	d
1	0	a	b	c	d	1

$\sim$	0	a	b	c	d	1
0	1	0	0	0	0	0
a	0	1	d	d	d	d
b	0	d	1	d	d	d
c	0	d	d	1	d	d
d	0	d	d	d	1	1
1	0	d	d	d	1	1

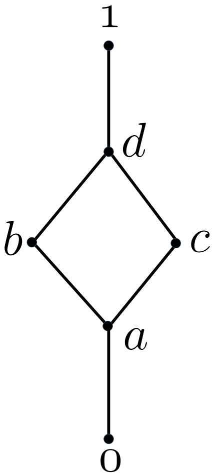
Examples of EQ-algebras — 6-element non-separated EQ-algebra



$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	a	a	a	b
c	0	0	a	0	a	c
d	0	0	a	a	a	d
1	0	a	b	c	d	1

$\sim$	0	a	b	c	d	1
0	1	0	0	0	0	0
a	0	1	d	d	d	d
b	0	d	1	d	d	d
c	0	d	d	1	d	d
d	0	d	d	d	1	1
1	0	d	d	d	1	1

Examples of EQ-algebras — 6-element non-separated EQ-algebra



$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	a	a	a	b
c	0	0	a	0	a	c
d	0	0	a	a	a	d
1	0	a	b	c	d	1

$\sim$	0	a	b	c	d	1
0	1	0	0	0	0	0
a	0	1	d	d	d	d
b	0	d	1	d	d	d
c	0	d	d	1	d	d
d	0	d	d	d	1	1
1	0	d	d	d	1	1

Examples of EQ-algebras — 6-element non-separated EQ-algebra

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	1	1	1	1
b	0	1	1	1	1	1
c	0	1	1	1	1	1
d	0	d	d	d	1	1
1	0	d	d	d	1	1

Not residuated:

$$0 = a \otimes d \leq 0 \quad \text{but} \quad a \not\leq d \rightarrow 0 = 0$$

Delta operation

In linearly ordered structure of truth values:

$$\Delta(a) = \begin{cases} \mathbf{1} & \text{if } a = \mathbf{1}, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

(determined algebraically in partially ordered structures)

All structures of truth values considered in FTT must contain the delta operation

Fuzzy equality

Given a set M

$$\doteq : M \times M \longrightarrow E$$

(i) Reflexivity: $[m \doteq m] = \mathbf{1}$

(ii) Symmetry: $[m \doteq m'] = [m' \doteq m]$

(iii) Transitivity: $[m \doteq m'] \otimes [m' \doteq m''] \leq [m \doteq m'']$

Example

$M = \mathbb{R}$, $\mathcal{E}$ is Łukasiewicz MV-algebra

$$[m \doteq m'] = 0 \vee (1 - |m - m'|)$$

Fuzzy equality

Given a set M

$$\doteq : M \times M \longrightarrow E$$

(i) Reflexivity: $[m \doteq m] = \mathbf{1}$

(ii) Symmetry: $[m \doteq m'] = [m' \doteq m]$

(iii) Transitivity: $[m \doteq m'] \otimes [m' \doteq m''] \leq [m \doteq m'']$

Example

$M = \mathbb{R}$, $\mathcal{E}$ is Łukasiewicz MV-algebra

$$[m \doteq m'] = 0 \vee (1 - |m - m'|)$$

Fuzzy equality

Given a set M

$$\stackrel{\circ}{=}: M \times M \longrightarrow E$$

- **(i)** Reflexivity: $[m \stackrel{\circ}{=} m] = \mathbf{1}$
- **(ii)** Symmetry: $[m \stackrel{\circ}{=} m'] = [m' \stackrel{\circ}{=} m]$
- **(iii)** Transitivity: $[m \stackrel{\circ}{=} m'] \otimes [m' \stackrel{\circ}{=} m''] \leq [m \stackrel{\circ}{=} m'']$

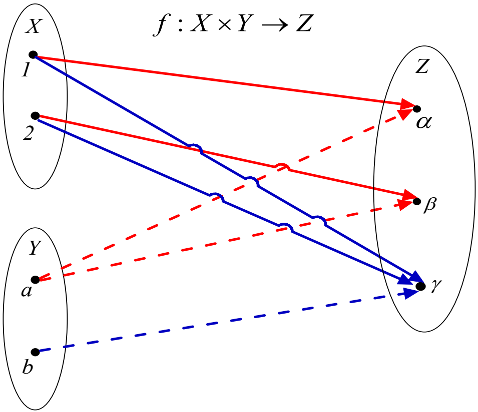
Example

$M = \mathbb{R}$, $\mathcal{E}$ is Łukasiewicz MV-algebra

$$[m \stackrel{\circ}{=} m'] = 0 \vee (1 - |m - m'|)$$

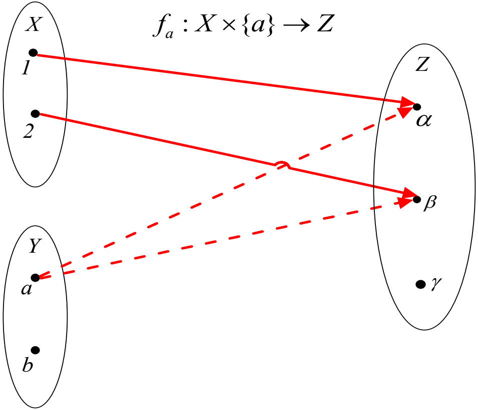
Function of n variables can be expressed using functions of one variable

M. Schönfinkel, H. Curry, G. Frege
(*currying*)



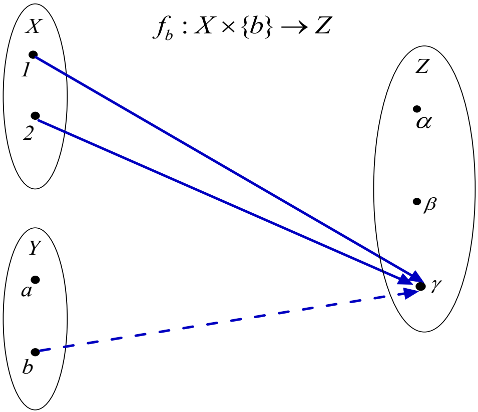
Function of n variables can be expressed using functions of one variable

M. Schönfinkel, H. Curry, G. Frege
(*currying*)



Function of n variables can be expressed using functions of one variable

M. Schönfinkel, H. Curry, G. Frege
(*currying*)



Types are indexes of specific sets

Elementary types:

(i) o (truth values) — $M_o = E$

(ii) ϵ (objects) — M_ϵ

Compound types:

Functions $f : M_\alpha \longrightarrow M_\beta$ form a subset $M_{\beta\alpha} \subseteq M_\beta^{M_\alpha}$

Types

(i) $\epsilon, o \in Types$,

(ii) If $\alpha, \beta \in Types$ then $(\alpha\beta) \in Types$.

Alternative notation

Write $\alpha \to \beta$ instead of $\beta\alpha$

IRAFM

Types are indexes of specific sets

Elementary types:

(i) o (truth values) — $M_o = E$

(ii) ϵ (objects) — M_ϵ

Compound types:

Functions $f : M_\alpha \longrightarrow M_\beta$ form a subset $M_{\beta\alpha} \subseteq M_\beta^{M_\alpha}$

Types

(i) $\epsilon, o \in Types$,

(ii) If $\alpha, \beta \in Types$ then $(\alpha\beta) \in Types$.

Alternative notation

Write $\alpha \to \beta$ *instead of* $\beta\alpha$

IRAFM

Types are indexes of specific sets

Elementary types:

(i) o (truth values) — $M_o = E$

(ii) ϵ (objects) — M_ϵ

Compound types:

Functions $f : M_\alpha \longrightarrow M_\beta$ form a subset $M_{\beta\alpha} \subseteq M_\beta^{M_\alpha}$

Types

(i) $\epsilon, o \in Types$,

(ii) If $\alpha, \beta \in Types$ then $(\alpha\beta) \in Types$.

Alternative notation

Write $\alpha \to \beta$ *instead of* $\beta\alpha$

IRAFM

Types are indexes of specific sets

Elementary types:

(i) o (truth values) — $M_o = E$

(ii) ϵ (objects) — M_ϵ

Compound types:

Functions $f : M_\alpha \longrightarrow M_\beta$ form a subset $M_{\beta\alpha} \subseteq M_\beta^{M_\alpha}$

Types

(i) $\epsilon, o \in$ *Types*,

(ii) If $\alpha, \beta \in$ *Types* then $(\alpha\beta) \in$ *Types*.

Alternative notation

Write $\alpha \to \beta$ instead of $\beta\alpha$

Types are indexes of specific sets

Elementary types:

(i) o (truth values) — $M_o = E$

(ii) ϵ (objects) — M_ϵ

Compound types:

Functions $f : M_\alpha \longrightarrow M_\beta$ form a subset $M_{\beta\alpha} \subseteq M_\beta^{M_\alpha}$

Types

(i) $\epsilon, o \in$ *Types*,

(ii) If $\alpha, \beta \in$ *Types* then $(\alpha\beta) \in$ *Types*.

Alternative notation

Write $\alpha \rightarrow \beta$ *instead of* $\beta\alpha$

General frame

$$\mathcal{M} = \langle \{M_\alpha, \mathring{=}_\alpha \mid \alpha \in Types\}, \mathcal{E}_\Delta \rangle$$

$$(M_o = \{a \mid a \in L\}, \sim) \qquad (M_\epsilon = \{m \mid \varphi(m)\}, =_\epsilon)$$

$$(M_{oo} \subseteq \{g_{oo} \mid g_{oo} : M_o \longrightarrow M_o\}, =_{oo})$$

$$(M_{o\epsilon} \subseteq \{f_{o\epsilon} \mid f_{o\epsilon} : M_\epsilon \longrightarrow M_o\}, =_{o\epsilon})$$

$$(M_{\epsilon\epsilon} \subseteq \{f_{\epsilon\epsilon} \mid f_{\epsilon\epsilon} : M_\epsilon \longrightarrow M_\epsilon\}, =_{\epsilon\epsilon}), \ldots$$

$$\vdots$$

$$(M_{\beta\alpha} \subseteq \{f_{\beta\alpha} \mid f_{\beta\alpha} : M_\alpha \longrightarrow M_\beta\}, =_{\beta\alpha})$$

$$\vdots$$

IRAFM

General frame

$$\mathcal{M} = \langle \{M_\alpha, \mathring{=}_\alpha \mid \alpha \in Types\}, \mathcal{E}_\Delta \rangle$$

$$(M_o = \{a \mid a \in L\}, \sim) \qquad (M_\epsilon = \{m \mid \varphi(m)\}, =_\epsilon)$$

$$(M_{oo} \subseteq \{g_{oo} \mid g_{oo} : M_o \longrightarrow M_o\}, =_{oo})$$

$$(M_{o\epsilon} \subseteq \{f_{o\epsilon} \mid f_{o\epsilon} : M_\epsilon \longrightarrow M_o\}, =_{o\epsilon})$$

$$(M_{\epsilon\epsilon} \subseteq \{f_{\epsilon\epsilon} \mid f_{\epsilon\epsilon} : M_\epsilon \longrightarrow M_\epsilon\}, =_{\epsilon\epsilon}), \ldots$$

$$\vdots$$

$$(M_{\beta\alpha} \subseteq \{f_{\beta\alpha} \mid f_{\beta\alpha} : M_\alpha \longrightarrow M_\beta\}, =_{\beta\alpha})$$

$$\vdots$$

General frame

$$\mathcal{M} = \langle \{M_\alpha, \mathring{=}_\alpha \mid \alpha \in Types\}, \mathcal{E}_\Delta \rangle$$

$$(M_o = \{a \mid a \in L\}, \sim) \qquad (M_\epsilon = \{m \mid \varphi(m)\}, =_\epsilon)$$

. . . .

$$(M_{oo} \subseteq \{g_{oo} \mid g_{oo} : M_o \longrightarrow M_o\}, =_{oo})$$

$$(M_{o\epsilon} \subseteq \{f_{o\epsilon} \mid f_{o\epsilon} : M_\epsilon \longrightarrow M_o\}, =_{o\epsilon})$$

$$(M_{\epsilon\epsilon} \subseteq \{f_{\epsilon\epsilon} \mid f_{\epsilon\epsilon} : M_\epsilon \longrightarrow M_\epsilon\}, =_{\epsilon\epsilon}), \ldots$$

. . . .

$$\vdots$$

. . . .

$$(M_{\beta\alpha} \subseteq \{f_{\beta\alpha} \mid f_{\beta\alpha} : M_\alpha \longrightarrow M_\beta\}, =_{\beta\alpha})$$

. . . .

$$\vdots$$

Formulas

Formulas

(i) Variables x_α and constants c_α are formulas of type α.

(ii) If $B_{\beta\alpha}$ and A_α are formulas then $(B_{\beta\alpha}A_\alpha)$ is a formula of type β.

(iii) If A_β is a formula and $x_\alpha \in J$ a variable then $\lambda x_\alpha\, A_\beta$ is a formula of type $\beta\alpha$.

Formulas A_o are propositions

Alternative notation

1. *Write $A : \alpha$ instead of A_α*
2. *Formulas are also called lambda terms*

Formulas

Formulas

(i) Variables x_α and constants c_α are formulas of type α.

(ii) If $B_{\beta\alpha}$ and A_α are formulas then $(B_{\beta\alpha}A_\alpha)$ is a formula of type β.

(iii) If A_β is a formula and $x_\alpha \in J$ a variable then $\lambda x_\alpha\, A_\beta$ is a formula of type $\beta\alpha$.

Formulas A_o are propositions

Alternative notation

1. *Write $A : \alpha$ instead of A_α*
2. *Formulas are also called lambda terms*

Fuzzy equality

Special formula (constant)

$$\mathbf{E}_{(o\alpha)\alpha}$$

Definition of fuzzy equality

$$\equiv \;:=\; \lambda x_\alpha \lambda y_\alpha (\mathbf{E}_{(o\alpha)\alpha}\, y_\alpha) x_\alpha .$$

$A_\alpha \equiv B_\alpha$ is a *formula of type o*

All connectives in FTT are formulas

Fuzzy equality

Special formula (constant)

$$\mathbf{E}_{(o\alpha)\alpha}$$

Definition of fuzzy equality

$$\equiv \;:=\; \lambda x_\alpha \lambda y_\alpha (\mathbf{E}_{(o\alpha)\alpha}\, y_\alpha) x_\alpha .$$

$A_\alpha \equiv B_\alpha$ is a *formula of type o*

All connectives in FTT are formulas

Truth values in FTT

Truth values should form either of:

1. a complete linearly ordered IMTL$_\Delta$-algebra
2. linearly ordered Łukasiewicz$_\Delta$-algebra
3. linearly ordered BL$_\Delta$-algebra
4. linearly ordered EQ$_\Delta$-algebra or IEQ$_\Delta$-algebra

Truth values in FTT

Truth values should form either of:

1. a complete linearly ordered IMTL$_\Delta$-algebra
2. linearly ordered Łukasiewicz$_\Delta$-algebra
3. linearly ordered BL$_\Delta$-algebra
4. linearly ordered EQ$_\Delta$-algebra or IEQ$_\Delta$-algebra

Truth values in FTT

Truth values should form either of:

1. a complete linearly ordered IMTL$_\Delta$-algebra
2. linearly ordered Łukasiewicz$_\Delta$-algebra
3. linearly ordered BL$_\Delta$-algebra
4. linearly ordered EQ$_\Delta$-algebra or IEQ$_\Delta$-algebra

IRAFM

Truth values in FTT

Truth values should form either of:

1. a complete linearly ordered IMTL$_\Delta$-algebra
2. linearly ordered Łukasiewicz$_\Delta$-algebra
3. linearly ordered BL$_\Delta$-algebra
4. linearly ordered EQ$_\Delta$-algebra or IEQ$_\Delta$-algebra

Interpretation of formulas

$$\mathcal{M}^{\mathcal{E}}(A_{\beta\alpha}) \in M_{\beta\alpha}$$

Example (interpretation)

$\mathcal{M}^{\mathcal{E}}(A_o) \in L$ is a truth value
$\mathcal{M}^{\mathcal{E}}(A_{o\epsilon})$ is fuzzy set in M_ϵ
$\mathcal{M}^{\mathcal{E}}(A_{(o\epsilon)\epsilon})$ is fuzzy relation on M_ϵ
$\mathcal{M}^{\mathcal{E}}(A_{(oo)\epsilon})$ is fuzzy set of type 2
$\mathcal{M}^{\mathcal{E}}(A_{\epsilon\epsilon})$ is function on objects

Logical axioms of IMTL-FTT (IMTL$_\Delta$ algebra)

Fundamental axioms

(FT1) $\Delta(x_\alpha \equiv y_\alpha) \Rightarrow (f_{\beta\alpha}\, x_\alpha \equiv f_{\beta\alpha}\, y_\alpha)$

(FT2$_1$) $(\forall x_\alpha)(f_{\beta\alpha}\, x_\alpha \equiv g_{\beta\alpha}\, x_\alpha) \Rightarrow (f_{\beta\alpha} \equiv g_{\beta\alpha})$

(FT2$_2$) $(f_{\beta\alpha} \equiv g_{\beta\alpha}) \Rightarrow (f_{\beta\alpha}\, x_\alpha \equiv g_{\beta\alpha}\, x_\alpha)$

(FT3) $(\lambda x_\alpha B_\beta)A_\alpha \equiv C_\beta$

where C_β is obtained from B_β by replacing all free occurrences of x_α in it by A_α (*lambda conversion*)

(FT4) $(x_\epsilon \equiv y_\epsilon) \Rightarrow ((y_\epsilon \equiv z_\epsilon) \Rightarrow (x_\epsilon \equiv z_\epsilon))$

Logical axioms of IMTL-FTT (IMTL$_\Delta$ algebra)

Equivalence axioms

(FT6) $(x_o \equiv y_o) \equiv ((x_o \Rightarrow y_o) \wedge (y_o \Rightarrow x_o))$

(FT7) $(A_o \equiv \top) \equiv A_o$

Implication axioms

(FT8) $(A_o \Rightarrow B_o) \Rightarrow ((B_o \Rightarrow C_o) \Rightarrow (A_o \Rightarrow C_o))$

(FT9) $(A_o \Rightarrow (B_o \Rightarrow C_o)) \equiv (B_o \Rightarrow (A_o \Rightarrow C_o))$

(FT10) $((A_o \Rightarrow B_o) \Rightarrow C_o) \Rightarrow (((B_o \Rightarrow A_o) \Rightarrow C_o) \Rightarrow C_o)$

(FT11) $(\neg B_o \Rightarrow \neg A_o) \equiv (A_o \Rightarrow B_o)$

Logical axioms of IMTL-FTT (IMTL$_\Delta$ algebra)

Conjunction axioms

(FT12) $A_o \wedge B_o \equiv B_o \wedge A_o$

(FT13) $A_o \wedge B_o \Rightarrow A_o$

(FT14) $(A_o \wedge B_o) \wedge C_o \equiv A_o \wedge (B_o \wedge C_o)$

Delta axioms

(FT5) $(g_{oo}(\mathbf{\Delta} x_o) \wedge g_{oo}(\neg \mathbf{\Delta} x_o)) \equiv (\forall y_o) g_{oo}(\mathbf{\Delta} y_o)$

(FT15) $\mathbf{\Delta}(A_o \wedge B_o) \equiv \mathbf{\Delta} A_o \wedge \mathbf{\Delta} B_o$

(FT16) $\mathbf{\Delta}(A_o \vee B_o) \Rightarrow \mathbf{\Delta} A_o \vee \mathbf{\Delta} B_o$

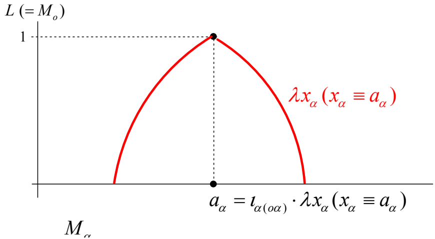
Logical axioms of IMTL-FTT (IMTL$_\Delta$ algebra)

Predicate axioms

(FT17) $(\forall x_\alpha)(A_o \Rightarrow B_o) \Rightarrow (A_o \Rightarrow (\forall x_\alpha)B_o)$, x_α is not free in A_o

Axiom of descriptions

(FT18) $\iota_{\epsilon(o\epsilon)}(\mathbf{E}_{(o\epsilon)\epsilon}\, y_\epsilon) \equiv y_\epsilon$



Inference rules and provability

Inference rules

(Rule R) *Let* $A_\alpha \equiv A'_\alpha$ *and* $B \in Form_o$. *Then infer* B' *where* B' *comes form* B *by replacing one occurrence of* A_α, *which is not preceded by* λ, *by* A'_α.

(Rule (N)) *Let* $A_o \in Form_o$ *be a formula. Then from* A_o *infer* ΔA_o.

Theory

- Theory T of FTT is a set of formulas of type o
- $T \vdash A$ — as usual

Inference rules and provability

Inference rules

(Rule R) *Let $A_\alpha \equiv A'_\alpha$ and $B \in Form_o$. Then infer B' where B' comes form B by replacing one occurrence of A_α, which is not preceded by λ, by A'_α.*

(Rule (N)) *Let $A_o \in Form_o$ be a formula. Then from A_o infer ΔA_o.*

Theory

- Theory T of FTT is a set of formulas of type o
- $T \vdash A$ — as usual

- Representation of *truth* and *falsity*

$$\top := (\lambda x_o\, x_o \equiv \lambda x_o\, x_o) \qquad \bot := (\lambda x_o\, x_o \equiv \lambda x_o\, \top)$$

- *Modus ponens* and *generalization* are derived rules

Theorem (Deduction theorem)

Let T be a theory, $A_o \in Form_o$ a formula. Then

$$T \cup \{A_o\} \vdash B_o \quad \textit{iff} \quad T \vdash \Delta A_o \Rightarrow B_o$$

holds for every formula $B_o \in Form_o$.

- Representation of *truth* and *falsity*

$$\top := (\lambda x_o\, x_o \equiv \lambda x_o\, x_o) \qquad \bot := (\lambda x_o\, x_o \equiv \lambda x_o\, \top)$$

- *Modus ponens* and *generalization* are derived rules

Theorem (Deduction theorem)

Let T be a theory, $A_o \in Form_o$ a formula. Then

$$T \cup \{A_o\} \vdash B_o \quad \text{iff} \quad T \vdash \Delta A_o \Rightarrow B_o$$

holds for every formula $B_o \in Form_o$.

- Representation of *truth* and *falsity*

$$\top := (\lambda x_o \, x_o \equiv \lambda x_o \, x_o) \qquad \bot := (\lambda x_o \, x_o \equiv \lambda x_o \top)$$

- *Modus ponens* and *generalization* are derived rules

Theorem (Deduction theorem)

Let T be a theory, $A_o \in Form_o$ a formula. Then

$$T \cup \{A_o\} \vdash B_o \quad \textit{iff} \quad T \vdash \mathbf{\Delta} A_o \Rightarrow B_o$$

holds for every formula $B_o \in Form_o$.

Completeness

Theorem

(a) *A theory T of IMTL-FTT is consistent iff it has a general model $\mathcal{M}$.*

(b) *For every theory T of IMTL-FTT and a formula A_o*

$$T \vdash A_o \quad \text{iff} \quad T \models A_o.$$

Completeness

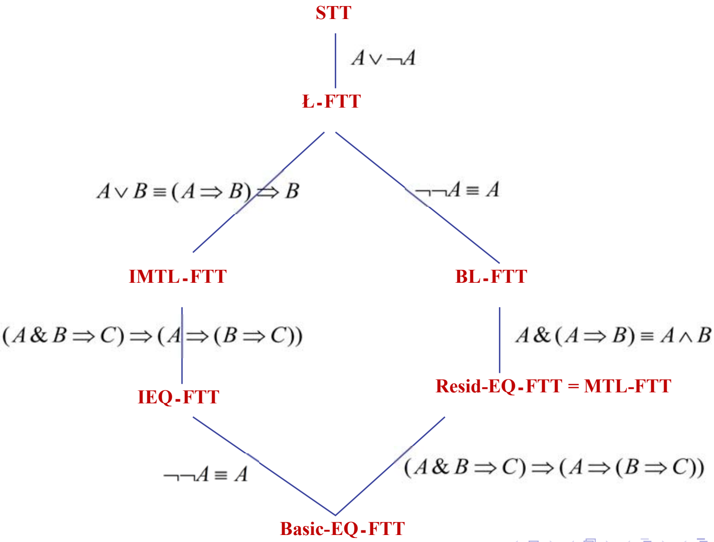
Theorem

(a) *A theory T of IMTL-FTT is consistent iff it has a general model $\mathcal{M}$.*

(b) *For every theory T of IMTL-FTT and a formula A_o*

$$T \vdash A_o \quad \textit{iff} \quad T \models A_o.$$

System of FTTs



Claim

All essential properties of vague predicates are formally expressible in FTT and so, they have a many-valued model

- Formalization of the sorites (falakros) paradoxes
- Various practical extensions of FTT that can be (and are) effectively implemented
  - (a) Theory of the meaning of evaluative linguistic expressions
  - (b) Theory of fuzzy/linguistic IF-THEN rules and learning
  - (c) Theory of intermediate quantifiers
  - (d) Theory of common sense reasoning

Claim

All essential properties of vague predicates are formally expressible in FTT and so, they have a many-valued model

- Formalization of the sorites (falakros) paradoxes
- Various practical extensions of FTT that can be (and are) effectively implemented
  - (a) Theory of the meaning of evaluative linguistic expressions
  - (b) Theory of fuzzy/linguistic IF-THEN rules and learning
  - (c) Theory of intermediate quantifiers
  - (d) Theory of common sense reasoning

Claim

All essential properties of vague predicates are formally expressible in FTT and so, they have a many-valued model

- Formalization of the sorites (falakros) paradoxes
- Various practical extensions of FTT that can be (and are) effectively implemented
  - **(a)** Theory of the meaning of evaluative linguistic expressions
  - **(b)** Theory of fuzzy/linguistic IF-THEN rules and learning
  - **(c)** Theory of intermediate quantifiers
  - **(d)** Theory of common sense reasoning

- *Higher-order fuzzy logic* (FTT) is a nice and transparent formal theory
- FTT provides means for characterization of difficult problems connected with the vagueness phenomenon
- FTT is highly expressive and has many useful virtues
- FTT has great potential for various kinds of applications

- *Higher-order fuzzy logic* (FTT) is a nice and transparent formal theory
- FTT provides means for characterization of difficult problems connected with the vagueness phenomenon
- FTT is highly expressive and has many useful virtues
- FTT has great potential for various kinds of applications

- *Higher-order fuzzy logic* (FTT) is a nice and transparent formal theory
- FTT provides means for characterization of difficult problems connected with the vagueness phenomenon
- FTT is highly expressive and has many useful virtues
- FTT has great potential for various kinds of applications

- *Higher-order fuzzy logic* (FTT) is a nice and transparent formal theory
- FTT provides means for characterization of difficult problems connected with the vagueness phenomenon
- FTT is highly expressive and has many useful virtues
- FTT has great potential for various kinds of applications

- P. Andrews, *An Introduction to Mathematical Logic and Type Theory: To Truth Through Proof*, Kluwer, Dordrecht, 2002.
- A. Church, *A formulation of the simple theory of types*, J. Symb. Logic 5 (1940) 56–68.
- W. M. Farmer, *The seven virtues of simple type theory*, Journal of Applied Logic 6 (2008) 267–286.
- L. Henkin, *Completeness in the theory of types*, J. Symb. Logic 15 (1950) 81–91.
- L. Henkin, *A theory of propositional types*, Fundamenta Math. 52 (1963) 323–344.
- V. Novák, *On fuzzy type theory*, Fuzzy Sets and Systems 149 (2005) 235–273.
- V. Novák, *EQ-algebra-based fuzzy type theory and its extensions*, Logic Journal of the IGPL (to appear).
- V. Novák, *A comprehensive theory of trichotomous evaluative linguistic expressions*, Fuzzy Sets and Systems 159 (22) (2008) 2939-2969.
- V. Novák, S. Lehmke, *Logical structure of fuzzy IF-THEN rules*. Fuzzy Sets and Systems 157(2006), 20032029.
- L. Běhounek, P. Cintula, *Fuzzy class theory*, Fuzzy Sets and Systems 154 (2005) 34–55.