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International Conference
Olomoucian Days of Applied Mathematics

ODAM 2011

Department of Mathematical analysis
and Applications of Mathematics
Faculty of Science
Palacký University Olomouc

F-Transform – a New Paradigm in Fuzzy Modeling

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Outline

This talk is about

- **Soft computing** approach to **mathematical** modeling
- Applications to **Image/Data processing**

What is Soft Computing ?

- **Soft Computing (SC)** - a modern mathematical discipline that is based on the fuzzy set theory
- Distinguishing Feature of SC is **Interpretability** of its tools and results

Outline

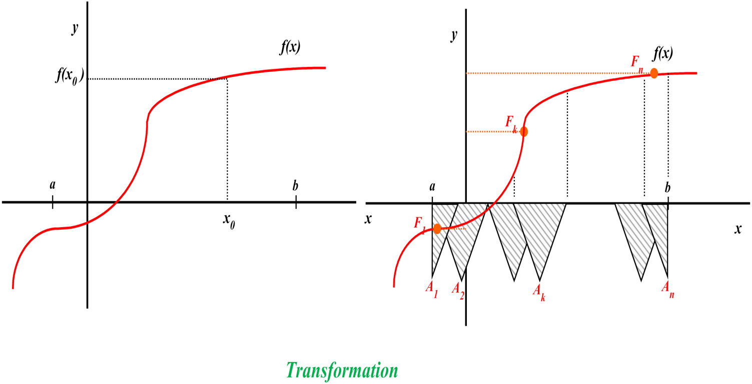
General Characterization of Fuzzy (F)-Transform

F-Transform is a universal method of soft computing

F-Transform

- stems from fuzzy IF-THEN rules
- uses fuzzy partition of a universe
- has two parts: direct and inverse
- explains how applications can be developed on its base

F-Transform Scheme



$$f: x \to f(x) \longrightarrow F_n[f]: A_i \to F_i,$$

F-Transform Prerequisites

- Interval $[a, b]$
- Fuzzy partition $A_1, \ldots, A_n$, $n \geq 2$, of $[a, b]$
- Set of integrable on $[a, b]$ functions with inner products

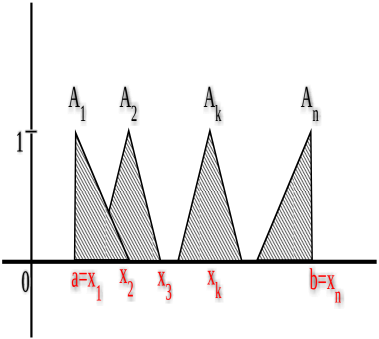
$$(f, g) = \int_a^b f(x)g(x)A_k(x)dx,\ k = 1, \ldots, n.$$

Outline

Fuzzy Partition of $[a, b]$

Fuzzy sets $A_1, \ldots, A_n$ with continuous membership functions form a **fuzzy partition** with nodes $\mathbf{x}_1, \ldots, \mathbf{x}_n$ if for each $k = 1 \ldots, n$

- $A_k(x_k) = 1$
- $A_k(x) = 0$ if $x \notin (x_{k-1}, x_{k+1})$
- $A_k(x) \nearrow$ on $[x_{k-1}, x_k]$
- $A_k(x) \searrow$ on $[x_k, x_{k+1}]$
- $\sum_{k=1}^{n} A_k(x) = 1, x \in [a, b]$



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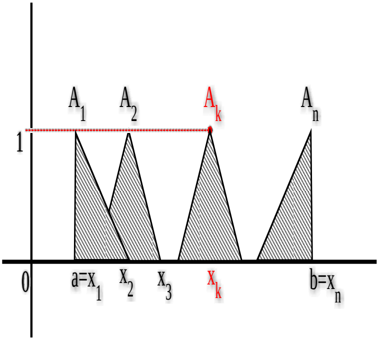

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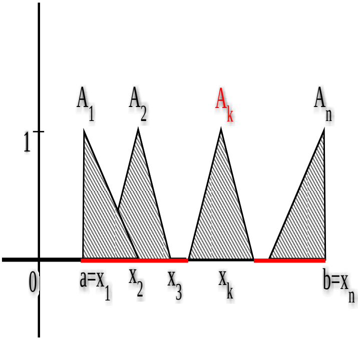

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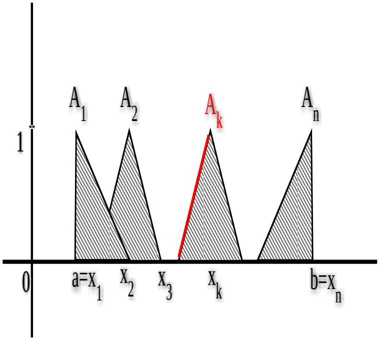

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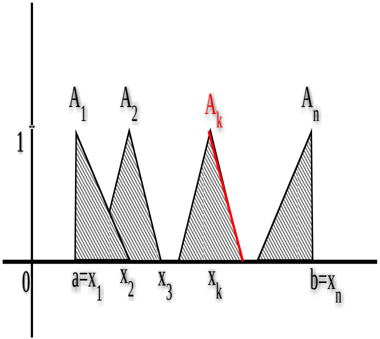

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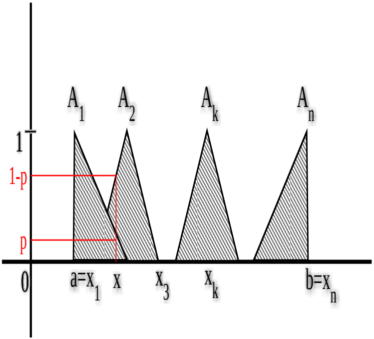
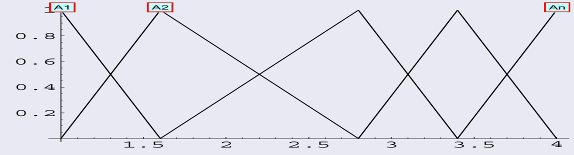


Various Fuzzy Partitions



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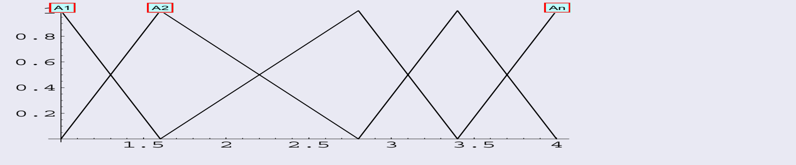

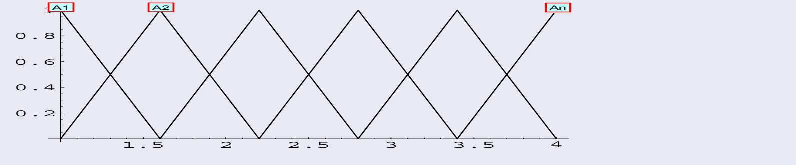

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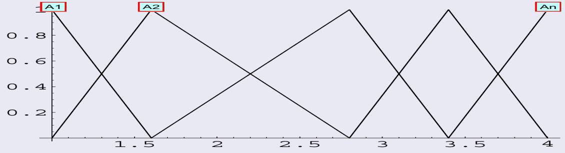
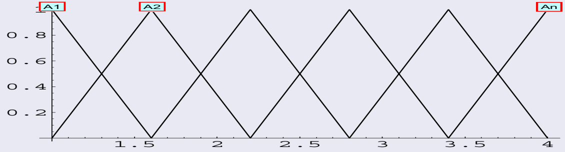
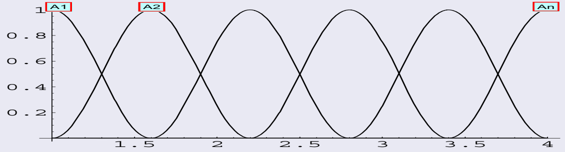






Fuzzy Partitions Formally

Triangular Partition

$$A_1(x) = \begin{cases} 1 - \frac{(x - x_1)}{h_1}, & x \in [x_1, x_2], \\ 0, & \text{otherwise}, \end{cases}$$

$$A_k(x) = \begin{cases} \frac{(x - x_{k-1})}{h_{k-1}}, & x \in [x_{k-1}, x_k], \\ 1 - \frac{(x - x_k)}{h_k}, & x \in [x_k, x_{k+1}], \\ 0, & \text{otherwise}, \end{cases}$$

$$A_n(x) = \begin{cases} \frac{(x - x_{n-1})}{h_{n-1}}, & x \in [x_{n-1}, x_n], \\ 0, & \text{otherwise}. \end{cases}$$

where $k = 2, \ldots n-1$, and $h_k = x_{k+1} - x_k$.

Fuzzy Partitions Formally

Uniform Sinusoidal Partition

$$A_1(x) = \begin{cases} 0.5(\cos \frac{\pi}{h}(x - x_1) + 1), & x \in [x_1, x_2], \\ 0, & \text{otherwise}, \end{cases}$$

$$A_k(x) = \begin{cases} 0.5(\cos \frac{\pi}{h}(x - x_k) + 1), & x \in [x_{k-1}, x_{k+1}], \\ 0, & \text{otherwise}, \end{cases}$$

$$A_n(x) = \begin{cases} 0.5(\cos \frac{\pi}{h}(x - x_n) + 1), & x \in [x_{n-1}, x_n], \\ 0, & \text{otherwise}. \end{cases}$$

where $k = 2, \ldots n-1$, and $h = \frac{x_n - x_1}{n-1}$.

Outline

F-Transform of function f. Integral Form

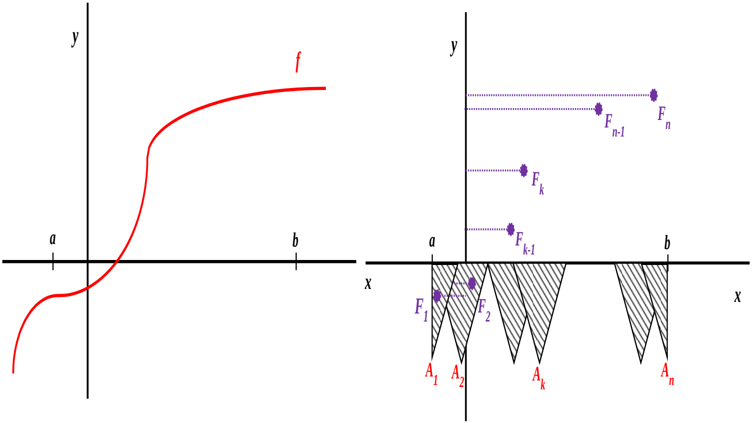
Definition

- Let f be defined and integrable on $[a, b]$
- A vector of real numbers $(F_1, \ldots, F_n)$ is the **F-transform** of f w.r.t. $A_1, \ldots, A_n$ if

$$F_k = \frac{\int_a^b f(x)A_k(x)dx}{\int_a^b A_k(x)dx}$$

Notation: $\mathbf{F}_n[f] = (F_1, \ldots, F_n)$

F-Transform. Illustration



$$\mathbf{F}_n[f] = (\mathbf{F}_1, \ldots, \mathbf{F}_n)$$

F-Transform of function f. Discrete Form

Definition

- Let f be given at points $x_1, \dots, x_l \in [a, b]$
- $\sum_{j=1}^{l} A_k(x_j) > 0, k = 1, \dots, n$
- A vector of real numbers $(F_1, \dots, F_n)$ is a **discrete F-transform** of f w.r.t. $A_1, \dots, A_n$ if

$$F_k = \frac{\sum_{j=1}^{l} f(x_j)A_k(x_j)}{\sum_{j=1}^{l} A_k(x_j)}$$

Outline

Main Properties of F-Transform

Best Approximation

Component F_k, $k = 1, \ldots, n$, **minimizes the following criterion**

$$\Phi_k(h) = \int_a^b (f(x) - h)^2 A_k(x) dx.$$

F_k is the **best approximation** of f on $[x_{k-1}, x_{k+1}]$ among all constant functions.

Main Properties of F-Transform

F-Transform of Constants

Component F_k, $k = 1, \ldots, n$, of a **constant function** f coincides with f.

Main Properties of F-Transform

Linearity

The F-transform of f is an **image of a linear mapping** $\mathbf{F}_n$, i.e. for all integrable f, h, and for all $\alpha, \beta \in \mathbb{R}$,

$$\mathbf{F}_n[\alpha f + \beta h] = \alpha \mathbf{F}_n[f] + \beta \mathbf{F}_n[h].$$

Outline

Inverse F-Transform

Definition

Let $(F_1, \ldots, F_n)$ be the F-transform of f w.r.t. $A_1, \ldots, A_n$. The following function

$$f_n(x) = \sum_{k=1}^{n} F_k A_k(x)$$

is called the **inverse F-transform** of f.

Inverse F-Transform. Uniform convergence

Uniform convergence

For a sequence $\{A_1^{(n)}, \dots, A_n^{(n)}\}$ of uniform partitions of $[a, b]$ the respective **sequence $\{f_n(x)\}$ of inverse F-transforms** of $f \in C[a, b]$ uniformly converges to f,

$$f_n(x) \underset{n \longrightarrow \infty}{\rightrightarrows} f(x).$$

F-Transform and Different Partitions

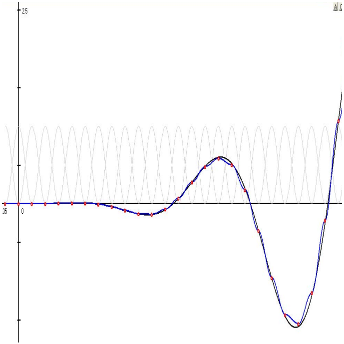
- $\{A'_1(x), \ldots, A'_n(x)\}$ $\{A''_1(x), \ldots, A''_n(x)\}$ – different fuzzy partitions of $[a, b]$.
- $\{f'_n(x)\}$, $\{f''_n(x)\}$ – the respective inverse F-transforms.

For all $x \in [a, b]$

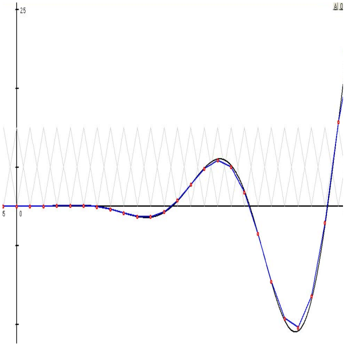
$$|f'_n(x) - f''_n(x)| \leq 4\omega(h, f).$$

F-Transform and Different Partitions

Cosine Shapes



Triangular Shapes



List of Applications

- Compression and Reconstruction of Images
- **Image Fusion**
- Fuzzy Initial Value Problem
- Numeric Solutions of ODE and PDE
- Smooth Logical Deduction
- Data Mining
- Time Series Analysis and Prediction

Outline

Fusion of Images

Description of a Problem

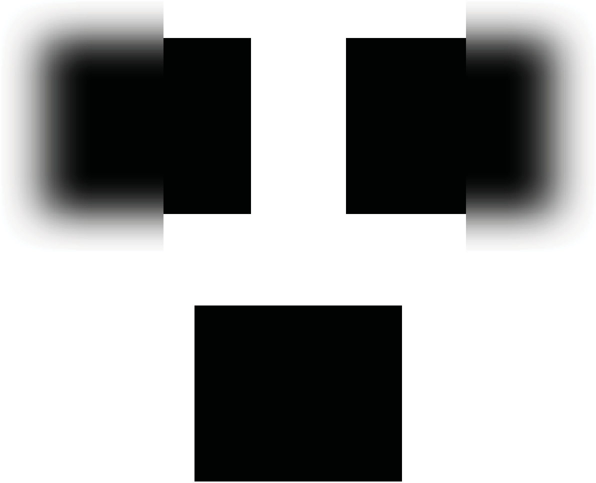
- Ideal image u – intensity function of two variables,
- $C_1, \ldots, C_N$ – acquired channels,
- $C_i(x,y) = D_i(u(x,y)) + E_i(x,y)$ – channel images (functions),
- Image Fusion – combining undistorted parts of each channel.

Examples of F-Transform Based Fusion

Fusion of a Blurred Picture

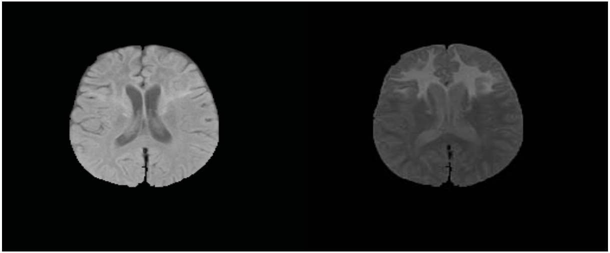
Examples of F-Transform Based Fusion

Fusion of a Blurred Picture



Examples of F-Transform Based Fusion

Image-Fusion Methods in Medical Diagnostics



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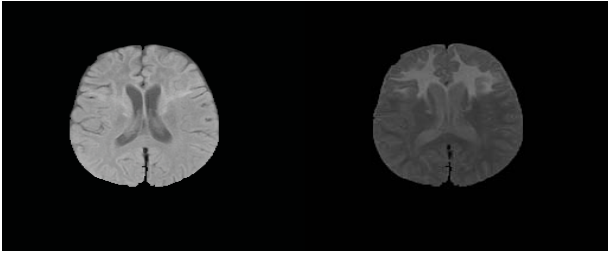
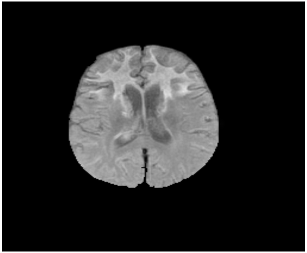




Examples of F-Transform Based Fusion

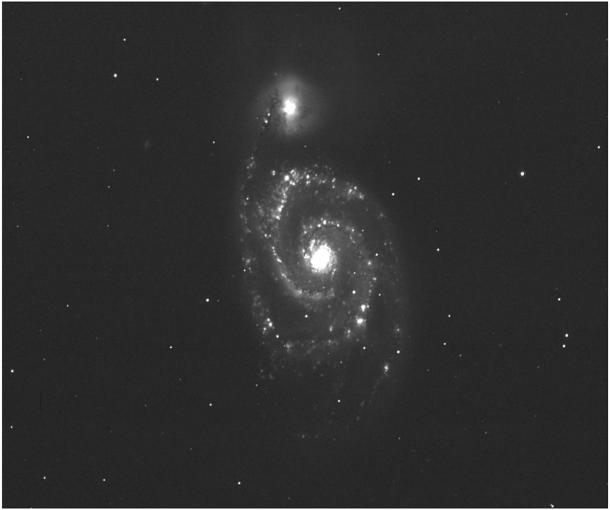
Galaxy M51: Source 1

Examples of F-Transform Based Fusion

Galaxy M51: Source 2

Examples of F-Transform Based Fusion

Galaxy M51: Source 3



Examples of F-Transform Based Fusion

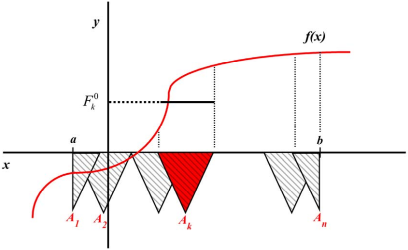
Galaxy M51: Fusion of Three Sources

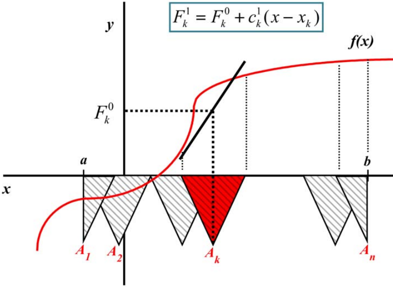
Why We Need a Higher Order F-Transform

Several Reasons in Favor of a Higher Order F-Transform

- higher smoothness,
- more meaningful parameters,
- better approximation by a corresponding component,
- faster convergence of the inverse F-transform.

How the F-Transform can be generalized





$$F_k = F_k^0 = \frac{\int_a^b (f(x) \bullet 1) A_k(x)dx}{\int_a^b A_k(x)dx}$$

$$c_k^1 = \frac{\int_a^b f(x) \bullet (x - x_k) A_k(x)dx}{\int_a^b (x - x_k)^2 A_k(x)dx}$$

Spaces of Integrable Functions

Spaces $L_2(A_1), \dots, L_2(A_n)$

- $A_1, \dots, A_n$ – a fuzzy partition of $[a, b]$;
- $L_2(A_k)$, $1 \le k \le n$ – a set of functions $f : [x_{k-1}, x_{k+1}] \longrightarrow \mathbb{R}$ for which the following integral

$$\int_{x_{k-1}}^{x_{k+1}} f^2(x) A_k(x) dx$$

exists.

Hilbert spaces

Let k be a fixed integer from $\{1, \ldots, n\}$.

- $(f, g)_k = \frac{1}{s_k} \int_{x_{k-1}}^{x_{k+1}} f(x)g(x)A_k(x)dx$ – **inner product** of f and g in $L_2(A_k)$, where
- $$s_k = \int_{x_{k-1}}^{x_{k+1}} A_k(x)dx, \quad k = 1, \ldots, n.$$
- $\|f\|_k = \sqrt{(f, f)_k}$ – **norm** in $L_2(A_k)$,

$L_2(A_k)$ is a **Hilbert space**

Orthogonal Polynomials in $L_2(A_k)$

Let

- $\{1, x, x^2, \dots x^m\}$, $m \geq 0$, – linearly independent system of polynomials restricted to $[x_{k-1}, x_{k+1}]$,
- $P_k^0, P_k^1, P_k^2, \dots, P_k^m$ – are **orthogonal polynomials** in $L_2(A_k)$ obtained from $\{1, x, x^2, \dots x^m\}$ by the Gram-Schmidt process.

Subspace $L_2^m(A_k)$

$L_2^m(A_k)$–linear subspace of $L_2(A_k)$ with the basis $P_k^0, \ldots, P_k^m$.

$L_2^m(A_k)$ consists of all polynomials of a degree $l \leq m$ restricted to $[x_{k-1}, x_{k+1}]$.

Obviously,

$$L_2^0(A_k) \subset L_2^1(A_k) \subset \ldots L_2^m(A_k) \subset \ldots$$

F^m-transform

Definition

Let

- $f : [a, b] \longrightarrow \mathbb{R}$ – function from $L_2(A_1) \cap \cdots \cap L_2(A_n)$,
- F_k^m, $m \geq 0$, – k-th orthogonal projection of $f|_{[x_{k-1}, x_{k+1}]}$ on $L_2^m(A_k)$, $k = 1, \ldots, n$.

F^m-transform of f with respect to $A_1, \ldots, A_n$ is

$$F^m[f] = (F_1^m, \ldots, F_n^m).$$

F_k^m is called the k^{th} F^m-transform **component** of f.

Representation of F^m-transform Components

Let

$$F_k^m = c_{k,0}P_k^0 + c_{k,1}P_k^1 + \cdots + c_{k,m}P_k^m.$$

Then

$$c_{k,i} = \frac{(f, P_k^i)_k}{(P_k^i, P_k^i)_k} = \frac{\int_a^b f(x)P_k^i(x)A_k(x)dx}{\int_a^b P_k^i(x)P_k^i(x)A_k(x)dx}, \quad i = 0, \dots, m.$$

Four Main Properties of the F^m-transform

(A) F^m-transform component F_k^m, $k = 1, \ldots, n$, **minimizes the following functional**

$$\Phi_k(h) = \int_a^b (f(x) - h(x))^2 A_k(x) dx,$$

defined on $L_2^m(A_k)$.

(B) F^m-transform component F_k^m, $k = 1, \ldots, n$, of a polynomial f of a degree $l \leq m$, restricted to $[x_{k-1}, x_{k+1}]$, coincides with f.

(C)

$$F_k^m = F_k^{m-1} + c_{k,m} P_k^m, \quad m = 1, 2, \ldots.$$

(D) Mapping $\mathcal{F}^m : f \mapsto (F_1^m, \ldots, F_n^m)$ is **linear**, i.e. for all $f, h \in L_2(A_1, \ldots, A_n)$, and for all $\alpha, \beta \in \mathbb{R}$,

$$\mathcal{F}^m(\alpha f + \beta h) = \alpha \mathcal{F}^m(f) + \beta \mathcal{F}^m(h).$$

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$$\Phi_k(h) = \int_a^b (f(x) - h(x))^2 A_k(x)dx,$$

defined on $L_2^m(A_k)$.

(B) F^m-transform component F_k^m, $k = 1, \ldots, n$, of a polynomial f of a degree $l \leq m$, restricted to $[x_{k-1}, x_{k+1}]$, coincides with f.

(C)

$$F_k^m = F_k^{m-1} + c_{k,m}P_k^m, \quad m = 1, 2, \ldots.$$

(D) Mapping $\mathcal{F}^m : f \mapsto (F_1^m, \ldots, F_n^m)$ is **linear**, i.e. for all $f, h \in L_2(A_1, \ldots, A_n)$, and for all $\alpha, \beta \in \mathbb{R}$,

$$\mathcal{F}^m(\alpha f + \beta h) = \alpha\mathcal{F}^m(f) + \beta\mathcal{F}^m(h).$$

Relationship between F-Transform and F^m-Transform Components

Let

- $(F_1^m, \ldots, F_n^m), m \geq 0$, – F^m-transform of f with respect to $A_1, \ldots, A_n$, and
- $F_k^m = c_{k,0}P_k^0 + c_{k,1}P_k^1 + \cdots + c_{k,m}P_k^m$, $k = 1, \ldots, n$.

Then

$(c_{1,0}, \ldots, c_{n,0})$ is the F-transform of f with respect to $A_1, \ldots, A_n$.

Approximation of f by F^m-transform Components

The theorem below shows that the **bigger** is the value of m, the better is the quality of approximation of $f|_{[x_{k-1},x_{k+1}]}$ by F_k^m.

Theorem

Let $m \geq 0$, and let polynomials F_k^m, F_k^{m+1} be orthogonal projections of $f \in L_2(A_k)$ on $L_2^m(A_k)$ and $L_2^{m+1}(A_k)$, respectively. Then

$$\|f|_{[x_{k-1},x_{k+1}]} - F_k^{m+1}\|_k \leq \|f|_{[x_{k-1},x_{k+1}]} - F_k^m\|_k.$$

F^1-Transform

Theorem

The following vector of linear functions

$$F^1[f] = (c_{1,0} + c_{1,1}(x - x_1), \dots, c_{n,0} + c_{n,1}(x - x_n))$$

is the F^1-transform of f with respect to $A_1, \dots, A_n$, where for every $k = 1, \dots, n$,

$$c_{k,0} = \frac{\int_{x_{k-1}}^{x_{k+1}} f(x)A_k(x)dx}{hs_0}, \tag{1}$$

$$c_{k,1} = \frac{\int_{x_{k-1}}^{x_{k+1}} f(x)(x - x_k)A_k(x)dx}{\int_{x_{k-1}}^{x_{k+1}} (x - x_k)^2 A_k(x)dx}, \tag{2}$$

and $s_0 = \int_{-1}^{1} A_0(x)dx$.

Characterization of F^1-transform components

Theorem

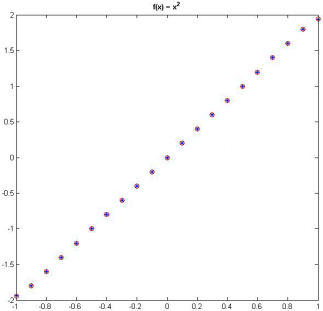
Let

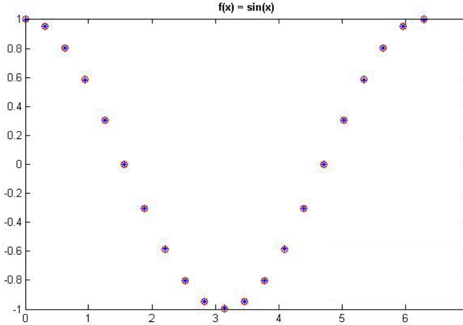
- f and A_k, $k = 1, \dots, n$, be four times continuously differentiable on $[a, b]$,
- $F^1[f] = (c_{1,0} + c_{1,1}(x - x_1), \dots, c_{n,0} + c_{n,1}(x - x_n))$ be the F^1-transform of f with respect to $A_1, \dots, A_n$.

Then

$$c_{k,1} = f'(x_k) + O(h) \quad k = 1, \dots, n.$$

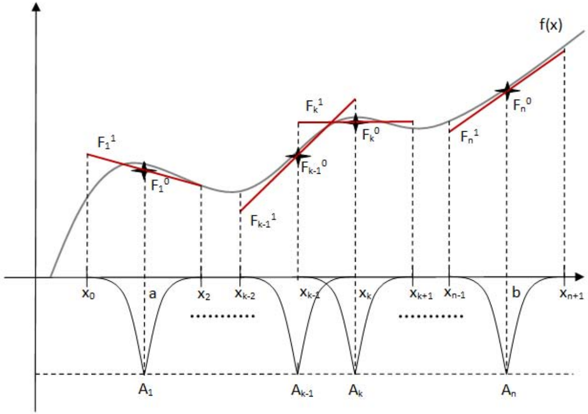
Illustration





Left. Coefficients $c_{k,1}$ of the F^1-transform of function x^2 on $[-1, 1]$ lie on the line $y = 2x$.
Right. Coefficients $c_{k,1}$ of the F^1-transform of function $sin(x)$ on $[0, 2\pi]$ lie on the line $y = cos(x)$.

Illustration



Function f and its F^1-transform components $F_1^1, \ldots, F_k^1, \ldots, F_n^1$.

F^1-transform in $L_2(A_k)$ with triangular A_k

Theorem

Let

- $A_1, \ldots, A_n$ – h-uniform triangular shaped partition of $[a, b]$ with generating function $A_0 = 1 - |x|$.

Then coefficients $c_{k,0}$ and $c_{k,1}$ in the representation of $F^1[f]$ are as follows:

$$c_{k,0} = \frac{\int_{x_{k-1}}^{x_{k+1}} f(x)A_k(x)dx}{h},$$

$$c_{k,1} = \frac{12 \int_{x_{k-1}}^{x_{k+1}} f(x)(x - x_k)A_k(x)dx}{h^3}.$$

Inverse F^m-transform

Definition

Let $(F_1^m, \dots, F_n^m)$, $m \geq 0$, be the F^m-transform of $f : [a, b] \longrightarrow \mathbb{R}$ with respect to $A_1, \dots, A_n$.
Then the following function $f_{F,n}^m : [a, b] \longrightarrow \mathbb{R}$

$$f_{F,n}^m(x) = \sum_{k=1}^{n} F_k^m A_k(x)$$

is called **the inverse F^m-transform** of f (with respect to $(F_1^m, \dots, F_n^m)$ and $A_1, \dots, A_n$).

$$f_{F,n}^m(x) = f_{F,n}^{m-1}(x) + \sum_{k=1}^{n} c_{k,m} P_k^m(x) A_k(x), \quad x \in [a, b],\ m \geq 1$$

Inverse F^m-transform as an Inverse Mapping

Theorem

Let

- $A_1, \ldots, A_n$ be an h-uniform partition of $[a, b]$,
- for all $x \in [a+h, b-h]$,

$$\sum_{i=1}^{n} A_i(x) = 1,$$

- p be a polynomial on $[a, b]$ of a degree $0 \le l \le m$.

Then the inverse F^m-transform $p^m_{F,n}$ of p coincides with p on $[a+h, b-h]$, i.e.

$$p^m_{F,n}(x) = p(x), \quad x \in [a+h, b-h].$$

Approximation by the inverse F^m-transform

Theorem

Let

- $A_1, \dots, A_n$ be an h-uniform Ruspini partition of $[a+h, b-h]$,
- functions f and A_k, $k = 1, \dots, n$, be four times continuously differentiable on $[a, b]$,
- $f^m_{F,n}$ be the inverse F^m-transform of f, $m \geq 1$.

Then

$$\int_{a+h}^{b-h} |f(x) - f^m_{F,n}(x)| dx \leq O(h^2),$$

or $\|f(x) - f^m_{F,n}\|_{L_1} \leq O(h^2)$ where L_1 is a normed space on $[a+h, b-h]$.

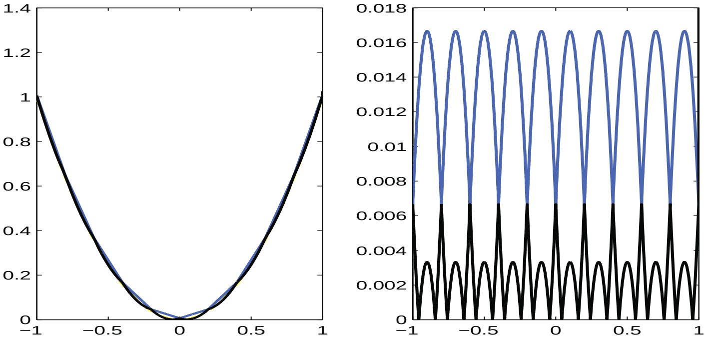
Uniform Convergence

Corollary

Under the assumptions of the Theorem above, the sequence $\{f^m_{F,n}\}$, $m \geq 1$, of the inverse F^m-transforms of f uniformly converges to f in the normed space L_1 on $[a+h, b-h]$, i.e.

$$\|f - f^m_{F,n}\|_{L_1} \rightrightarrows_n 0.$$

Illustration



Left. The function x^2 and its inverse F^0 (blue line) and F^1 (black line) transforms.
Right. Graphs of the error functions. Maximal errors of approximation are: 0.017 (the inverse F^0-transform) and 0.0062 (the inverse F^1-transform).

Conclusion

Our purpose was to develop:

- A universal technique for mathematical modeling
- Easy-to-explain theory
- Sound mathematical backgrounds
- Convincing applications