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OP Vzdělávání
pro konkurenceschopnost

INVESTICE
DO ROZVOJE
VZDĚLÁVÁNÍ

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ODAM 2013

Department of Mathematical analysis
and Applications of Mathematics
Faculty of Science
Palacký Univerzity Olomouc

Inconsistency evaluation in preference relations: a characterization based on metrics induced by a norm

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Outline

Pairwise Comparison Matrix PCM

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- Set of alternatives

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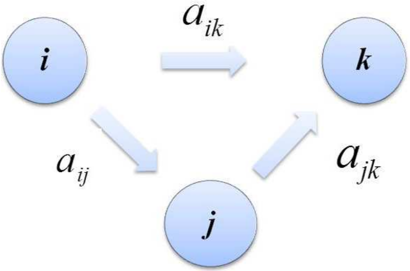

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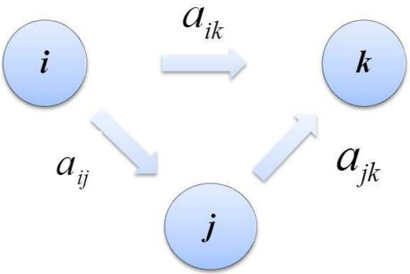

direct comparison a_{ik} confirm indirect comparison $a_{ij}a_{jk}$

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- Objective: to evaluate ‘how much’ the pairwise comparison matrix deviates from full consistency condition (1)

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- If judgments are far from consistency, it is likely that the decision maker expressed them with scarce competence and care

Some (in)consistency indices

- Consistency Index (CI); Saaty – 1977
- Geometric Consistency Index (GCI); Crawford, Williams – 1985
- Golden, Wang – 1989
- Koczkodaj – 1993
- Relative Error; Barzilai – 1998
- Shiraishi, Obata, Daigo – 1998
- Peláez , Lamata – 2003
- Harmonic Consistency Index (HCI); Stein, Mizzi – 2007
- Cavallo, D'Apuzzo – 2009
- Ramík , Korviny – 2010
- ....

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- But ... the notion of distance is too general
- Distances induced by norms have several interesting properties

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- multiplicative estimation of the degree of preference of x_i over x_j
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- additive estimation of the degree of preference of x_i over x_j
  $a_{ij} \in \mathbb{R}$
  $a_{ii} = 0 \; \forall i$
  $a_{ij} + a_{ji} = 0 \; \forall i, j$ additive reciprocity: $\mathbf{A}$ is skew–symmetric
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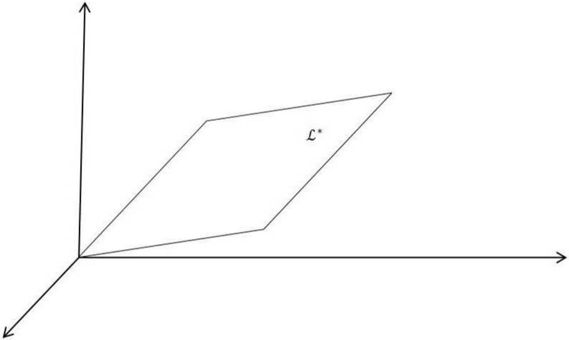
- $\mathcal{L}^* = \{\mathbf{A} \in \mathcal{L} |\ a_{ij} + a_{jk} = a_{ik}\ , \ \ i, j, k = 1, \ldots, n\}$
  the set of consistent matrices is a linear subspace of $\mathcal{L}$ (Koczkodaj and Orlowski, Comp. Math. Appl. 1997)

Linear subspace of consistent matrices



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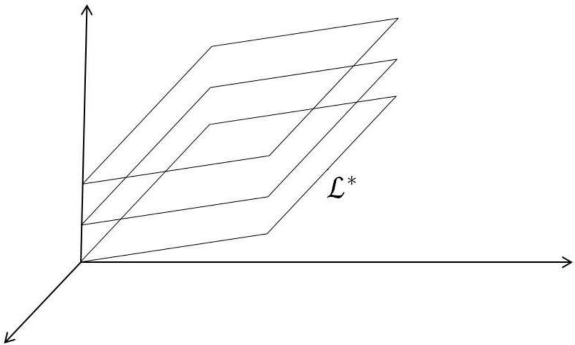

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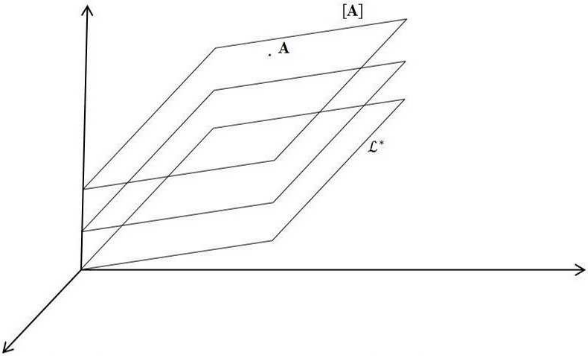

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- How to obtain (2)?

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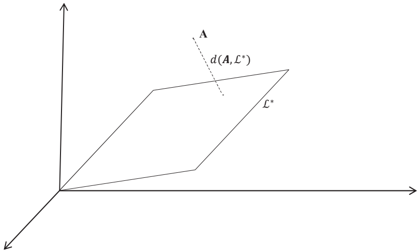

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(see theorem 2 in the following)

Characterizing properties of a norm

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1. $||\mathbf{x}|| \geq 0 \quad \forall \mathbf{x}$
2. $||\mathbf{x}|| = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$
3. $||k\mathbf{x}|| = |k| \, ||\mathbf{x}||$ for any scalar k (positive homogeneity)
4. $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{x}||$ (triangle inequality)

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If property 2 is removed, we obtain a seminorm

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2. $I_d(k\mathbf{A}) = |k| I_d(\mathbf{A}) \quad \forall \mathbf{A} \in \mathcal{L}, \quad \forall k \in \mathbb{R}$
3. $I_d(\mathbf{A} + \mathbf{A}') \leq I_d(\mathbf{A}) + I_d(\mathbf{A}') \quad \forall \mathbf{A}, \mathbf{A}' \in \mathcal{L}$.

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- If

$$d(\mathbf{A}, \mathcal{L}^*) = d(\mathbf{A}, \mathbf{A}^*),$$

i.e. $\mathbf{A}^* \in \mathcal{L}^*$ minimizes the distance of $\mathbf{A}$ from $\mathcal{L}^*$, then, by adding a consistent matrix $\mathbf{B} \in \mathcal{L}^*$ it is

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- If

$$d(\mathbf{A}, \mathcal{L}^*) = d(\mathbf{A}, \mathbf{A}^*),$$

i.e. $\mathbf{A}^* \in \mathcal{L}^*$ minimizes the distance of $\mathbf{A}$ from $\mathcal{L}^*$, then, by adding a consistent matrix $\mathbf{B} \in \mathcal{L}^*$ it is

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- semantic: inconsistency doesn't change by adding consistent preferences

Corollary

The function $I_d : \mathcal{L}/\mathcal{L}^* \rightarrow \mathbb{R}$, defined as follows

$$I_d([\mathbf{A}]) = I_d(\mathbf{A}), \quad \mathbf{A} \in [\mathbf{A}], \tag{5}$$

is a norm on $\mathcal{L}/\mathcal{L}^*$.

Corollary

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Only the equivalence class $\mathcal{L}^*$ of consistent matrices has zero – inconsistency

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where

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$\mathcal{L}^{\perp}$	is the set of totally inconsistent matrices

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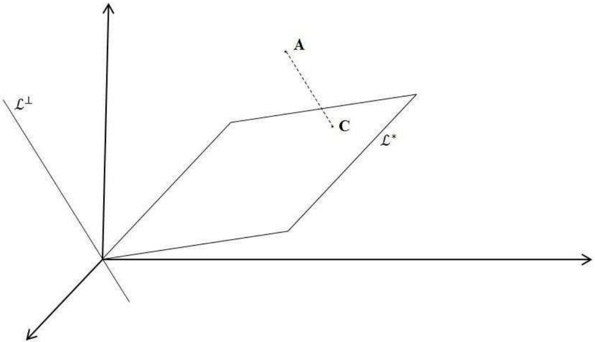
- Barzilai's orthogonal decomposition refers to the Euclidean norm

orthogonal projection



Characterizing properties for an inconsistency index

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If the norm defining $I_d(\mathbf{A}) = \min_{\mathbf{B}\in\mathcal{L}^*} ||\mathbf{A} - \mathbf{B}||$ is permutation invariant, then $I_d(\mathbf{A})$ satisfies the five characterizing properties introduced by *Brunelli and Fedrizzi – ISAHP 2011*

Property 1

Existence of a single value of $I(\mathbf{A})$ for every consistent matrix

$$\exists!\ \nu \in \mathbb{R} \mid I(\mathbf{A}) = \nu \Leftrightarrow \mathbf{A} \in \mathcal{L}^*$$

Property 1

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In our case it is

$$I_d(\mathbf{A}) = 0 \Leftrightarrow \mathbf{A} \in \mathcal{L}^*$$

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$$I(\mathbf{P}\mathbf{A}\mathbf{P}^T) = I(\mathbf{A})$$

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Since the norm $||\mathbf{A} - \mathbf{B}||$ is assumed to be permutation invariant, then property 2 is satisfied

Property 3

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- Note that $f(a_{ij}) = ka_{ij}$ is the unique transformation which preserves reciprocity

Property 3 – Formalization

For every PCM $\mathbf{A} = (a_{ij})$ and $k > 1$, it is

$$I(\mathbf{A}) \leq I(\hat{\mathbf{A}})$$

where $\hat{\mathbf{A}} = (ka_{ij})$

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- Consider a consistent PCM $\mathbf{A} \in \mathcal{L}^*$
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- By increasing or decreasing the value of a_{pq}, and modify its reciprocal a_{qp} accordingly, then the resulting matrix is not anymore consistent.

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The idea underlying (P4)

- the larger the change of a_{pq}, the more inconsistent becomes the matrix

Property 4 – Example

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modifying a consistent matrix $\mathbf{A} \in \mathcal{L}^*$

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then $I(\mathbf{A}) \leq I(\mathbf{A}') \leq I(\mathbf{A}'')$

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- Continuity of $I_d(\mathbf{A}) = \min_{\mathbf{B} \in \mathcal{L}^*} ||\mathbf{A} - \mathbf{B}||$ directly follows from continuity of each norm with respect to the induced topology

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- In the additive approach this corresponds to a linear combination

$$a_{ij}^G = \sum_{k=1}^{m} \lambda_k (a_{ij}^k) \quad i, j = 1, ..., n \tag{7}$$

$$\mathbf{A}^G = \sum_{k=1}^{m} \lambda_k \mathbf{A}^k \tag{8}$$

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- A weaker property has been studied and proved, for example, for Saaty's CI:

$$I(\mathbf{A}^G) \leq \max\{I(\mathbf{A}^1), ..., I(\mathbf{A}^m)\}$$

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different values of p produce different evaluations. Opportunity of suitable choices of p.

thanks

- Thanks for attention