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EVROPSKÁ UNIE

MINISTERSTVO ŠKOLSTVÍ,
MLÁDEŽE A TĚLOVÝCHOVY

OP Vzdělávání
pro konkurenceschopnost

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DO ROZVOJE
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and Applications of Mathematics
Faculty of Science
Palacký Univerzity Olomouc

Integrals: special functionals and optimization tool

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2 Special functionals

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Short history

1850 B.C., Egypt (Moscow Mathematical Papyrus, problem 14) volume of a frustrum of a square pyramide

370 B.C., Greece, EUDOXUS, exhaustion method

3^{rd} century B.C., Greece, ARCHIMEDES, parabols, circle

3^{rd} century A.C., China, LIU HUI, circle

5^{th} century A.C., China, ZU CHONG ZHI and ZU GENG, sphere

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1615, Austria (LINZ), KEPLER, volume of barels

1854, Germany (GÖTTINGEN), RIEMANN, habilitation thesis

1902, France (NANCY), LEBESGUE, doctoral thesis

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Special functionals

Special functionals

- Lebesgue integral $\sim$ linear operators

  Dunford & Schwartz 1958
- Choquet integral $\sim$ comonotone additivity

  Schmeidler 1986
- Sugeno integral $\sim$ comonotone maxitivity & min–homogeneity

  Marichal 2000

Special functionals

- Shilkret integral $\sim$ comonotone maxitivity & homogeneity

  Benvenuti et al. 2002
- concave integral $\sim$ concave functional, homogeneity, smallest with $I(m, 1_A) \geq m(A)$

  Lehrer 2009
- convex integral $\sim$ convex functional, homogeneity, greatest with $I(m, 1_A) \leq m(A)$

  Li et al. 2013

Decomposition integrals

Workers a, b, c

performance per hour in units

$m(\emptyset) = 0,\ m(a) = 2,\ m(b) = 3,\ m(c) = 4,\ m(a, b) = 7,\ m(b, c) = 5,\ m(a, c) = 4,\ m(a, b, c) = 8,$

capacity in hours $f(a) = 5,\ f(b) = 4,\ f(c) = 3.$

Determine the optimal total performance!

α) only one group can work a fixed time period

β) several disjoint groups can work (fixed time in each group may differ)

γ) one group starts to work, once a worker stop to work, he cannot start again

δ) there are no constraints

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$$V_\alpha = \max\{c \cdot m(A) \mid c \cdot 1_A \leq f\} = 4 \cdot 7 = 28$$

SHILKRET 1971

$$V_\beta = \max\left\{\sum c_i \cdot m(A_i) \mid \sum c_i \cdot 1_{A_i} \leq f, (A_i) \text{ system disj.}\right\} =$$

$$= 4 \cdot 7 + 3 \cdot 4 = 40$$

YANG 1983 (PAN–integral)

$$V_\gamma = \max\left\{\sum c_i \cdot m(A_i) \mid \sum c_i \cdot 1_{A_i} \leq f, (A_i) \text{ chain}\right\} =$$

$$= 3 \cdot 8 + 1 \cdot 7 + 1 \cdot 2 = 33$$

CHOQUET 1953

$$V_\delta = \max\left\{\sum c_i \cdot m(A_i) \mid \sum c_i \cdot 1_{A_i} \leq f\right\} = 4 \cdot 7 + 1 \cdot 2 + 3 \cdot 4 = 42$$

LEHRER 2009, CONCAVE INTEGRAL

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LEHRER 2009, CONCAVE INTEGRAL

$f(x) \sim$ minimal number of working hours

$m(A) \sim$ cost of group A work per hour

goal $\sim$ minimize the costs W!

$$W_\alpha = \min\{c \cdot m(A) \mid c \cdot 1_A \geq f\} = 5 \cdot 8 = 40$$

$$W_\beta = \min\left\{\sum c_i \cdot m(A_i) \mid \sum c_i \cdot 1_{A_i} \geq f, (A_i) \text{ disj. system}\right\} =$$

$$= 4 \cdot 5 + 5 \cdot 2 = 30$$

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CONVEX INTEGRAL

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CONVEX INTEGRAL

$$
\begin{array}{ccccccccc}
 & V_\delta = \text{Lehrer } 42 & & & W_\alpha \quad 40 & \\
\nearrow & & \nwarrow & \nearrow & & \nwarrow \\
V_\beta = \text{PAN } 40 & & W_\gamma = V\gamma = & \text{Choquet } 33 & & W_\beta \quad 30 \\
\nwarrow & & \nearrow & \nwarrow & & \nearrow \\
 & V_\alpha = \text{Shilkret } 28 & & & W_\delta = \text{ convex } 28 &
\end{array}
$$

Even & Lehrer decomposition integrals

$$V_{\mathcal{H}}(m,f) = \sup\left\{\sum c_i \cdot m(A_i) \mid \sum c_i \cdot 1_{A_i} \leq f, (A_i) \in \mathcal{H}\right\}$$

Mesiar, Li & Pap superdecomposition integrals

$$W_{\mathcal{H}}(m,f) = \inf\left\{\sum c_i \cdot m(A_i) \mid \sum c_i \cdot 1_{A_i} \geq f, (A_i) \in \mathcal{H}\right\}$$

Even & Lehrer
decomposition integrals

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Universal integrals

$$I : \bigcup_{(X,\mathcal{A})\in\mathcal{S}} \left(\mathcal{M}_{(X,\mathcal{A})} \times \mathcal{F}_{(X,\mathcal{A})}\right) \to [0,\infty]$$

$\mathcal{S}$ the class of all measurable spaces
$\mathcal{M}_{(X,\mathcal{A})}$ all monotone measures on $(X,\mathcal{A})$
$\mathcal{F}_{(X,\mathcal{A})}$ all non–negative measurable functions $f : X \to [0,\infty]$

(U/1) I is increasing in both coordinates

(U/2) there is pseudo–multiplication $\otimes : [0, \infty]^2 \to [0, \infty]$ such that $I(m, c1_E) = c \otimes m(E)$

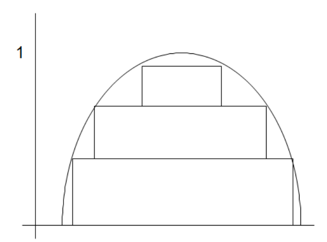
(U/3) $I(m_1, f_1) = I(m_2, f_2)$ whenever $m_1(f_1 \geq x) = m_2(f_2 \geq x)$ for all $x \in (0, \infty]$

If we constraint to $[0,1]$ case ("fuzzy"), we suppose $\otimes : [0,1]^2 \to [0,1]$ is a semicopula, and always $m(X) = 1$

Choquet 1953 (Šipoš 1979, but also Vitali 1925)

$$I_{Ch}(m,f) = \sup\left\{\sum c_i m\left(E_i\right) \mid \sum c_i \cdot 1_{E_i} \leq f,\ \left(E_i\right)\ \text{chain}\right\}$$

Universal integral



PAN–integral Yang 1983

á la Lebesgue, m need not be σ–additive

$$I_Y(m,f) = \sup\left\{\sum c_i m(E_i) \mid \sum c_i \cdot 1_{E_i} \le f,\ (E_i) \text{ disjoint}\right\}$$

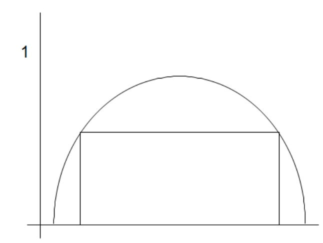
NOT a universal integral

Riemann	E_i intervals, m Lebesgue measure
Lebesgue	m σ–additive measure

Shilkret 1971

$$I_{Sh}(m,f) = \sup\{c \cdot m(E) \mid c \cdot 1_E \leq f\}$$

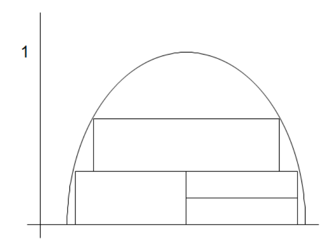
Universal integral



Lehrer 2009 (concave integral)

$$I_L(m, f) = \sup\left\{\sum c_i m(E_i) \mid \sum c_i \cdot 1_{E_i} \leq f\right\}$$

Not a universal integral



Defect of some integrals – they do not allow to recover original measure m!

PAN–integral

$$I_Y(m, 1_E) = m(E) \text{ for } \forall E \in \mathcal{A}$$

only if m is superadditive, $m(E \cup F) \geq m(E) + m(F), \quad E \cap F = \emptyset$

concave integral

$$I_L(m, 1_E) = m(E) \text{ for } \forall E \in \mathcal{A}$$

only if m is TB (totally balanced);
(supermodularity $m(A \cup B) + m(A \cap B) \geq m(A) + m(B)$ of m is enough!)

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Mesiar & Stupňanová

I is decomposition & universal integral iff $I \in \{I_{(1)}, I_{(2)}, \cdots, I_{(\infty)}\}$

$$I_{(n)}(m,f) = \sup\left\{\sum_{i=1}^{n} c_i \cdot m(A_i) \mid \sum c_i \cdot 1_{A_i} \le f, (A_i)_{i=1}^{n} \text{ is chain}\right\}$$

$$I_{(1)} \le I_{(2)} \le \cdots I_{(\infty)} = \sup I_{(n)}$$

$I_{(1)}$ SHILKRET $\qquad$ $I_{(\infty)}$ CHOQUET

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+ and · can be modified into $\oplus$ and $\odot$

For example,

$$x \oplus_p y = (x^p + y^p)^{\frac{1}{p}}\,, \quad p > 0$$

$$x \odot y = xy$$

pseudo–concave integral (Jun Li, R. Mesiar & E. Pap, 2011)

$$I_L^{\oplus_p,\odot}(m,f) = \sup\left\{\bigoplus_p (c_i \odot m(E_i)) \mid \bigoplus_p b(c_i,E_i) \le f\right\}$$

where $b(c,E)(x) = \begin{cases} c & \text{if } x \in E, \\ 0 & \text{else} \end{cases}$

qualitatively nothing new up to an isomorphism

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$\oplus = \vee$ (sup)

Working on $[0,1]$, considering $\odot$ any semicopula (1 is neutral element)

all 4 approaches are equivalent, yielding unique integral

$$I^{\oplus,\odot}(m,f) = \sup\{c \odot m(E) \mid b(c,E) \le f\} =$$

$$= \sup\{t \odot m(f \ge t) \mid t \in [0,1]\}$$

$I^{\vee,\wedge}$ SUGENO 1974

$I^{\vee,\cdot}$ SHILKRET 1971

$I^{\vee,T}$ WEBER 1986

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MUROFUSHI & SUGENO 1991,
fuzzy t–conorm integral (on [0, 1])

1. type $I^{\vee,\odot}$
2. type á la distorted Choquet

$$I_m^{h,k,g}(f) = h^{(-1)}\left(Ch_{g\circ m}(k\circ f)\right)$$

$g, h, k : [0,1] \to [0,\infty]$ additive
generators of t–conorms;
$h = g \sim$ strict t–conorm S
m is S–additive $\Rightarrow$ Weber integral (1984)

MESIAR 1996,
Choquet–like integral (on $[0, \infty]$)

1. type $I^{\vee,\otimes}$
2. type $I_m^g(f) = g^{-1}\left(Ch_{g \circ m}(g \circ f)\right)$
   $g : [0, \infty] \to [0, \infty]$ automorphism

Universal integrals based on copulas, on [0, 1]
KLEMENT, MESIAR, PAP 2004, 2010

$$C : [0,1]^2 \to [0,1]$$

$$C(0,x) = C(x,0) = 0$$

$$C(1,x) = C(x,1) = x$$

$$C((x_1,y_1) \vee (x_2,y_2)) + C((x_1,y_1) \wedge (x_2,y_2)) \geq C(x_1,y_1) + C(x_2,y_2)$$

$$C \longleftrightarrow P_C \text{ on } \mathcal{B}\left([0,1]^2\right),$$

$$P_C([a,b] \times [0,1]) = P_C([0,1] \times [a,b]) = b - a$$

$$I_C(m,f) = P_C\left(\{(u,v) \in [0,1]^2 \mid v \leq m(f \geq u)\}\right)$$

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$$C((x_1,y_1) \vee (x_2,y_2)) + C((x_1,y_1) \wedge (x_2,y_2)) \geq C(x_1,y_1) + C(x_2,y_2)$$

$$C \longleftrightarrow P_C \text{ on } \mathcal{B}\left([0,1]^2\right),$$

$$P_C([a,b] \times [0,1]) = P_C([0,1] \times [a,b]) = b - a$$

$$I_C(m,f) = P_C\left(\{(u,v) \in [0,1]^2 \mid v \leq m(f \geq u)\}\right)$$

Universal integrals based on copulas, on [0, 1]
KLEMENT, MESIAR, PAP 2004, 2010

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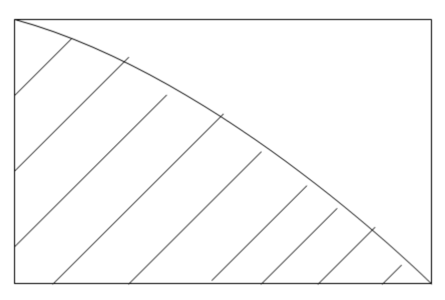

Figure: graph of the function $m(f \geq u)$

$$I_{\Pi} = I_C \qquad \text{Choquet}$$

$$I_{Min} = I^{\vee,\wedge} \qquad \text{Sugeno}$$

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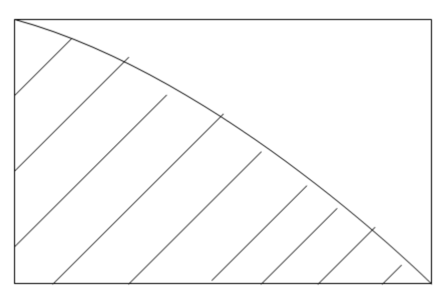

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decomposition & universal integrals based on copulas

$$I_{C,(n)}(m,f) = \sup\left\{\sum_{i=1}^{n}\left(C(c_1+\cdots+c_i, m(f\geq c_1+\cdots+c_i))-\right.\right.$$

$$\left.\left.-C(c_1+\cdots+c_{i-1}, m(f\geq c_1+\cdots+c_i))\right)\,|c_1,\cdots,c_n\geq 0\right\}$$

$$I_{C,(1)}\leq I_{C,(2)}\leq\cdots\leq I_{C,(\infty)}=\sup I_{C,(n)}\leq I_C$$

Fuzzy measure–based integral, on $[0, 1]$
Klement, Mesiar, Pap 2004

μ fuzzy measure on $\mathcal{B}\left([0,1]^2\right)$,
$\mu\left([0,c] \times [0,1]\right) = \mu\left([0,1] \times [0,c]\right) = c$

$$I_\mu(m,f) = \mu\left(\{(u,v) \in [0,1]^2 | v \leq m(f \leq u)\}\right)$$

$\mu(E) = \sup\left(uv | (u,v) \in E\right) \to$ SHILKRET

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on $[0, \infty]$ similar integrals

one application:
h–index, q–index

$$H = I^{\vee,\wedge}(m, f) \quad Q = I^{\vee,A}(m, \sqrt{f})$$

X all publications of an author
$m(E) = card\ E$
$f(i) = c_i$ number of citations of publication i

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1. Kolesárová A; Komorníková M: Triangular norm-based iterative compensatory operators. FUZZY SETS AND SYSTEMS 104 (1999) 109-120 (**38**)
2. Calvo T; Kolesárová A; Komorníková M; et al.: Aggregation operators: Properties, classes and construction methods. In: AGGREGATION OPERATORS: NEW TRENDS AND APPLICATIONS Book Series: Studies in Fuzziness and Soft Computing 97, pp. 3-104 Published: 2002 (**21**)
3. Komorník J.; Komorníková M.; Mesiar R.; et al.:Comparison of forecasting performance of nonlinear models of hydrological time series. PHYSICS AND CHEMISTRY OF THE EARTH 31 (2006)1127-1145 (**12**)
4. Komorníková M: Aggregation operators and additive generators. INTERNATIONAL JOURNAL OF UNCERTAINTY FUZZINESS AND KNOWLEDGE-BASED SYSTEMS 9 (2001) 205-215 (**11**)
5. Komorníková M.; Szolgay J.; Svetlíková D.; et al.: A hybrid modelling framework for forecasting monthly reservoir inflows. JOURNAL OF HYDROLOGY AND HYDROMECHANICS 56 (2008) 145-162 (**5**)
6. Hanus J; Komorníková M; Mináriková J: Influence of boxing materials on the properties of different paper items stored inside. RESTAURATOR-INTERNATIONAL JOURNAL FOR THE PRESERVATION OF LIBRARY AND ARCHIVAL MATERIAL 16 (1995) 94-208 (**4**)

h = 5, q = 3

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Thanks for your attention!