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INVESTICE
DO ROZVOJE
VZDĚLÁVÁNÍ

Streamlining the Applied Mathematics Studies
at Faculty of Science of Palacký University in Olomouc
CZ.1.07/2.2.00/15.0243

International Conference
Olomoucian Days of Applied Mathematics

ODAM 2013

Department of Mathematical analysis
and Applications of Mathematics

Faculty of Science

Palacký Univerzity Olomouc

Application of binary relations to ranking of alternatives

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ODAM Olomouc, June 12-14, 2013

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Part 1: Pair-wise comparison relations

Motivation example 1

$X = \{x_1, x_2, x_3, x_4\}$ - 4 alternatives (cars)
given pair-wise comparison matrix:
e.g. comparison according to „design“

$$A = \begin{pmatrix} 1 & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} \\ 8 & 1 & 5 & 3 \\ 4 & \frac{1}{5} & 1 & 2 \\ 2 & \frac{1}{3} & \frac{1}{2} & 1 \end{pmatrix}$$

$$\begin{matrix} x_1 & x_2 & x_3 & x_4 \end{matrix}$$

x_1 Škoda-Fabia

x_2 Opel – Corsa

x_3 Fiat – Punto

x_4 Renault-Clio

Rank alternatives according to design!

AHP: evaluation on the Saaty´s scale 1 to 9

Motivation example 1:
Solution (by T. Saaty, AHP)

$\boldsymbol{A} > \mathbf{0}$, **spectral radius** (maximum eigenvalue)

$$\rho(\boldsymbol{A}) = 4.152$$

where $\boldsymbol{Aw} = \rho(\boldsymbol{A})\boldsymbol{w}$

Inconsistency index: $$I_{mc}(A) = \frac{\rho(A) - n}{n-1} = \frac{4.152 - 4}{4-1} = 0.051$$

Inconsistency ratio: $$CR_{mc}(A) = \frac{I_{mc}(A)}{RI_n} = \frac{0.051}{0.9} \cong 0.06 < 0.1$$

and it has a positive (real) eigenvector – **priority vector**

$$\boldsymbol{w} = (0.061, 0.603, 0.201, 0.134) \;\Rightarrow$$

rank of alternatives is: $$x_2 > x_3 > x_4 > x_1$$

Motivation example 2

$X = \{x_1, x_2, x_3, x_4\}$ - 4 alternatives – „cars“ – criterion: design

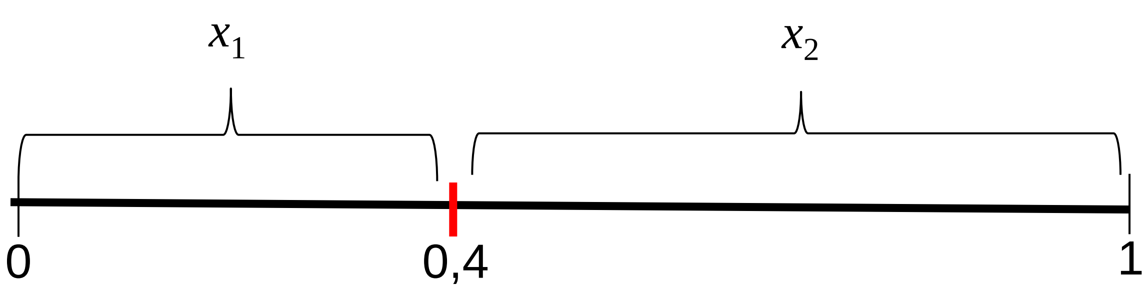
x_1 – „Škoda - Fabia“

x_2 – „Opel - Corsa“

x_3 – „Fiat - Punto“

x_4 – „Renault - Clio“

$$B = \begin{array}{cc} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} & \\ \begin{pmatrix} 0.5 & \boxed{0.4} & 0.6 & 0.7 \\ 0.6 & 0.5 & 0.6 & 0.9 \\ 0.4 & 0.4 & 0.5 & 0.5 \\ 0.3 & 0.1 & 0.5 & 0.5 \end{pmatrix} & \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \end{array}$$



General problem

$X = \{x_1, x_2,\ldots,x_n\}$ … set of n alternatives
(objects, persons, DM criteria,…)
Given a pair-wise comparison relation $\mathcal{A}$
Rank the alternatives (or, choose the best one)!

$\mathcal{A} \subset X\times X$ ...bin. relation on $X \rightarrow$(positive) matrix $A=\{a_{ij}\}$
a_{ij} … preference intensity of alternative x_i over x_j
$a_{ij} \in S$ - scale
e.g. $S = \{0,1\}$ – binary scale
or $\boxed{S =\{^1/_9, {^1/_8},\ldots,{^1/_2}, 1, 2, \ldots,8 ,9\} \text{ – AHP „Saaty“}}$
or $S = [^1/_\sigma ; \sigma]$, $\sigma > 1$ – interval scale
or $S = [0 ; 1]$ – interval scale
Properties of pair-wise comparison matrix $A=\{a_{ij}\}$:
Reciprocity and **consistency/transitivity**

Multiplicative matrix

$A = \{a_{ij}\} > 0$ is *multiplicative-reciprocal* (*m-reciprocal*), if

$$\boxed{a_{ij} . a_{ji} = 1 \quad \text{for all } i, j, \quad \text{or} \quad a_{ji} = \frac{1}{a_{ij}}}$$

$A = \{a_{ij}\}$ is *multiplicative-consistent* (*m-consistent*), if

$$\boxed{a_{ik} = a_{ij} . a_{jk} \quad \text{for all } i, j, k,}$$

$A = \{a_{ij}\}$ is *multiplicative-transitive* (*m-transitive*), if

$$\boxed{\frac{a_{ik}}{a_{ki}} = \frac{a_{ij}}{a_{ji}} \frac{a_{jk}}{a_{jk}} \quad \text{for all } i, j, k,}$$

Additive matrix

$A = \{a_{ij}\}$, $a_{ij} \in [0;1]$, is *additive-reciprocal (a-reciprocal)*, if

$$\boxed{a_{ij} + a_{ji} = 1 \quad \text{for all } i, j, \quad \text{or} \quad a_{ji} = 1 - a_{ij}}$$

$A = \{a_{ij}\}$ is *additive-transitive (a-transitive)* if

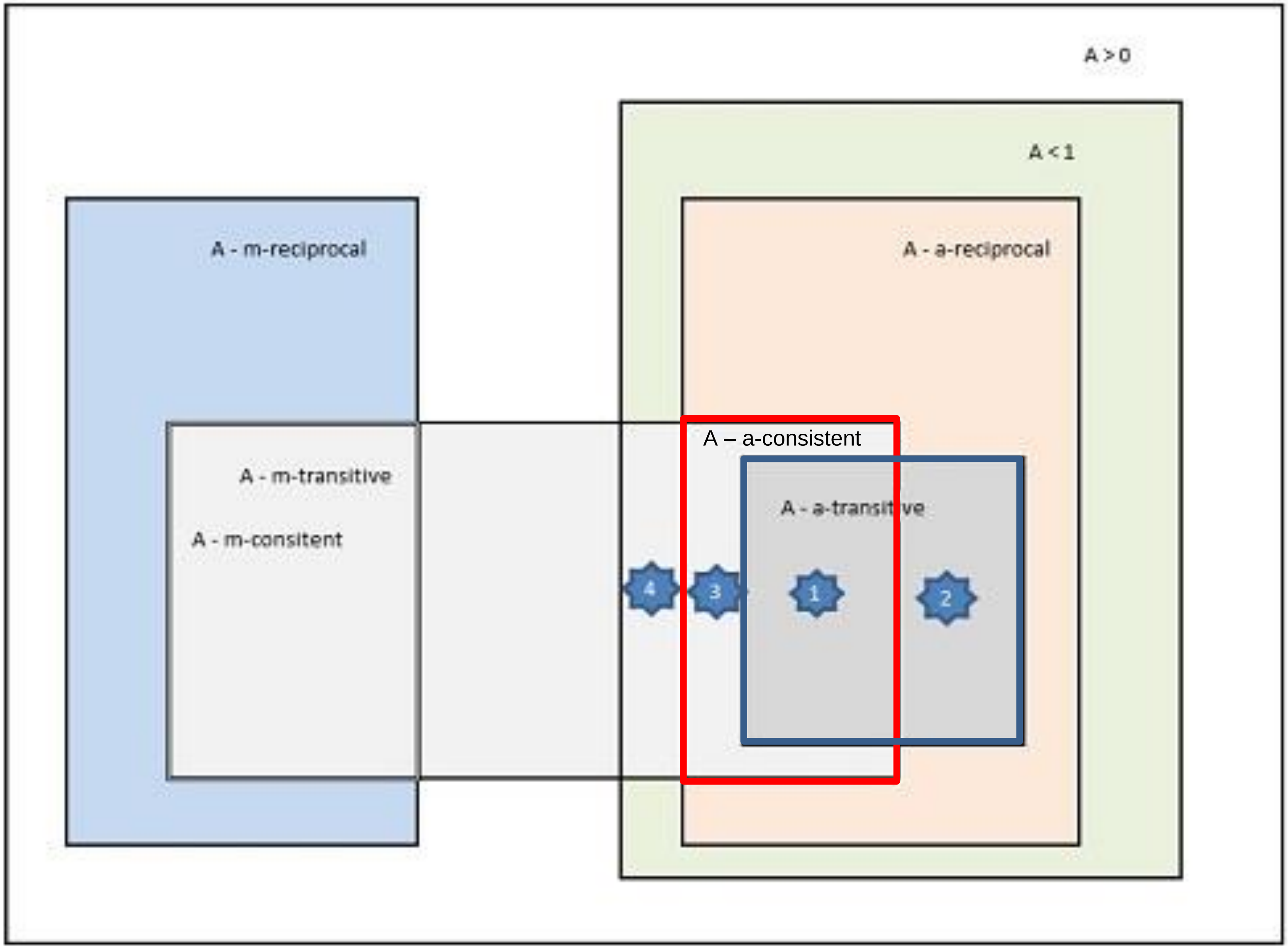
$$\boxed{(a_{ik} - 0.5) = (a_{ij} - 0.5) + (a_{jk} - 0.5) \quad \text{for all } i, j, k}$$

Remark 1. Elements a_{ij} = 0 and/or 1 are allowed !
Remark 2. If A is m-consistent, $a_{ij} \in [0;1]$, then A is a-transitive

Definition.

a-reciprocal and m-transitive *matrix* is called ***a-consistent***

Additive versus multiplacative matrices



Additive versus multiplacative matrices

Results: „*a*-consistency vers. *m*-consistency “
„*a*-transitivity vers. *m*-consistency “

Proposition 1. *Let* $A=\{a_{ij}\}$ *be an* ***a-reciprocal*** $n\text{x}n$ *matrix with* $0 < a_{ij} < 1$.

$A = \{a_{ij}\}$ *is m-transitive (i.e.* ***a-consistent****)* *iff* $B = \{\frac{a_{ij}}{1-a_{ij}}\}$ *is* ***m-consistent****.*

Proposition 2. *Let* $A = \{a_{ij}\}$ *be* $n\text{x}n$ *matrix*, $0 \le a_{ij} \le 1$ *for all i, j*
$\sigma > 1$

$A = \{a_{ij}\}$ ***a-transitive*** *matrix* *iff* $C = \{\sigma^{2a_{ij}-1}\}$

is ***m-consistent****.*

Additive versus multiplacative matrices

Results: „*a*-consistency vers. *m*-consistency "
„*a*-transitivity vers. *m*-consistency "

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$\sigma > 1$

$A=\{a_{ij}\}$ ***a-transitive*** *matrix* *iff* $C=\{\sigma^{2a_{ij}-1}\}$

is ***m-consistent****.*

Ranking alternativese and measuring consistency:
Perron-Frobenius theory - based approach

A - irreducible ***nonnegative*** matrix $(A \geq \mathbf{0})$
Then the spectral radius, $\rho(A)$, is a ***real*** eigenvalue, which has a ***positive*** (real) eigenvector $\boldsymbol{w} = (w_1, \ldots, w_n)$:

$$A\boldsymbol{w} = \rho(A)\boldsymbol{w}$$

This ***eigenvalue is simple*** and its ***eigenvector is unique*** up to a multiplicative constant.

Consequences for DM analysis:

- if A is positive ($A > 0$), then A is irreducible
- spectral radius, $\rho(A)$, is used for measuring consitency of A
- eigenvector $\boldsymbol{w} = (w_1, \ldots, w_n)$ – **priority vector** is used for ranking
- if A is m-reciprocal, then $\rho(A) \geq n$
- if A is m-reciprocal, then $\rho(A) = n$ iff A is m-consistent

m-consistency index

- $\boldsymbol{A}$ positive m-reciprocal nxn matrix $(\boldsymbol{A} > \boldsymbol{0})$
- ***m-consistency index*** I_{mc} is defined as

$$I_{mc}(A) = \frac{\rho(A) - n}{n-1}$$

- ***m-consistency ratio*** CR_{mc}:

$$CR_{mc}(A) = \frac{I_{mc}(A)}{RI_n}$$

- RI_n – random index (expected value over a large number of positive reciprocal matrices of dimension n)

Theorem 1. $\boldsymbol{A} = \{a_{ij}\}$ - positive m-reciprocal matrix.
$\boldsymbol{A}$ is m-consistent iff $I_{mc}(A) = 0$.

a-consistency index

- $\boldsymbol{A}$ - a-reciprocal nxn matrix $(0 < a_{ij} < 1)$
- ***a-consistency index*** I_{ac} : Proposition 1

$$I_{ac}(A) = I_{mc}(B) = \frac{\rho(B) - n}{n-1} \text{ where } B = \{\frac{a_{ij}}{1-a_{ij}}\}$$

- ***a-consistency ratio***: $$CR_{ac}(A) = \frac{I_{ac}(A)}{RI_n}$$
- ***priority vector***: $$\boldsymbol{w}^{ac} = (w_1^{ac}, w_2^{ac}, \ldots, w_n^{ac})$$
  where $$B\boldsymbol{w}^{mac} = \rho(B)\boldsymbol{w}^{ac}$$

Theorem 2. $\boldsymbol{A} = \{a_{ij}\}$ - positive a-reciprocal matrix
$\boldsymbol{A}$ is a-consistent iff $I_{ac}(A) = 0$

a-transitivity index

- $\boldsymbol{A}$ - a-reciprocal nxn matrix $(0 < a_{ij} < 1\,,\ \sigma > 1\ )$
- ***a-transitivity index*** I_{at}^{σ} :

Proposition 2

$$I_{at}^{\sigma}(A) = I_{mc}(B^{\sigma}) = \frac{\rho(B^{\sigma}) - n}{n-1} \quad \text{where } B^{\sigma} = \{\sigma^{2a_{ij}-1}\}$$

- ***a-transitivity ratio***: $$CR_{at}^{\sigma}(A) = \frac{I_{at}^{\sigma}(A)}{RI_n}$$
- ***priority vector***: $$\mathbf{w}^{at} = (w_1^{at}, w_2^{at}, \ldots, w_n^{at})$$

  where $$B^{\sigma}\mathbf{w}^{at} = \rho(B^{\sigma})\mathbf{w}^{at}$$

__Theorem 3.__ $\boldsymbol{A} = \{a_{ij}\}$ - positive a-reciprocal matrix

$\boldsymbol{A}$ is a-transitive iff $I_{at}^{\sigma}(A) = 0$

Other approaches

- **Valued preference modelling** (Fodor & Roubens, 1994) (valued preference relation = fuzzy pref. relation)
- **Similarity relations and valued orders** (Ovchinnikov, 1994)
- **Aggregation operator approach** (Dubois & Prade, 1984; Grabisch et al., 2009)
- **Ranking and choice procedures** (Fishburn, 1977; Orlovski, 1978)
- **Miscalaneous approaches**: AGREPREF, ELECTRE, PROMETHEE etc. (Roy, 1978; Vincke, 1988)

Part 2: Fuzzy preference matrix with missing elements

- Too many pair-wise comparisons: $N=n(n\text{-}1)/2$
- Missing information problem
- Acceptable number of comparisons is about n

  (i.e. dimension of the matrix)
- Eigenvector approach is not applicable
- We look for a **complete** matrix representation of incomplete matrix wich is as “close” to incomplete one as possible

Multiplicative/additive preference again…

$A = \{a_{ij}\}$ *is m-reciprocal* nxn *matrix,* $0 < a_{ij}$ *for all* i, j

Proposition 3.

$A = \{a_{ij}\}$ *is* ***m-****consistent iff there exists a positive vector of weights* $w=(w_1, w_2,..., w_n)$ *such that*

$$a_{ij} = \frac{w_i}{w_j} \quad \text{for all } i,j$$

$B = \{b_{ij}\}$ *is a-reciprocal* nxn *matrix,* $0 < b_{ij} < 1$ *for all* i, j

Proposition 4.

$B = \{b_{ij}\}$ *is* ***a-****consistent iff there exists a positive vector of weights* $v=(v_1, v_2,..., v_n)$ *such that*

$$b_{ij} = \frac{v_i}{v_i + v_j} \quad \text{for all } i,j$$

Proposition 5.

$B = \{b_{ij}\}$ *is* ***a-****transitive* ***iff*** *there exists a positive vector of weights* $u=(u_1, u_2,..., u_n)$ *such that*

$$b_{ij} = \tfrac{1}{2}(1 + nu_i - nu_j) \quad \text{for all } i,j$$

m/a-reciprocal matrix with missing elements: Notation

$$K = \overbrace{\{(1,2),(2,3),(3,4),\ldots,(n\text{-}1,n)\}}^{L\text{ - set of indices}} \cup \overbrace{\{(2,1),(3,2),(4,3),\ldots,(n,n\text{-}1)\}}^{L'\text{- symmetric set of indices}} \cup \overbrace{\{(1,1),(2,2),\ldots,(n,n)\}}^{D\text{ - diagonal}} =$$
$$= L \cup L' \cup D$$

$b_{ij} > 0$ for $(i,j) \in K$–set of evaluated indices, $K = L \cup L' \cup D$, $|L| \geq n\text{-}1$

$b_{ij} =$ "-" for $(i,j) \in I^2 - K$ – set of indices of missing elements

$$B(K) = \begin{pmatrix} 0.5 & b_{12} & - & \ldots & - \\ b_{21} & 0.5 & - & \ldots & b_{2n} \\ - & - & 0.5 & \ldots & b_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ - & b_{n2} & b_{n3} & \ldots & 0.5 \end{pmatrix}$$

$\boldsymbol{B}(K)$ - **fuzzy preference** matrix with missing elements with respect to K

Example 5: a-reciprocal matrix with missing elements

$$B(K) = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 0.5 & 0.9 & - & - \\ 0.1 & 0.5 & 0.8 & - \\ - & 0.2 & 0.5 & 0.6 \\ - & - & 0.4 & 0.5 \end{pmatrix} \end{array}$$

Incomplete matrix „upper diagonal“

$$B^{ac}(K) = \begin{pmatrix} 0.5 & 0.9 & \mathbf{0.96} & \mathbf{0.98} \\ 0.1 & 0.5 & 0.8 & \mathbf{0.86} \\ \mathbf{0.04} & 0.2 & 0.5 & 0.6 \\ \mathbf{0.02} & \mathbf{0.14} & 0.4 & 0.5 \end{pmatrix}$$

„Complete (ac-extension) matrix“

a-consistency based on deviations 1

$\boldsymbol{B}(K)$ - **fuzzy preference** matrix with missing elements with respect to $K = L \cup L' \cup D$, $|L| \geq n$-1

$\boldsymbol{v^*} = (v^*_1, v^*_2, \ldots, v^*_n)$ - **optimal solution** of optimization problem:

(P_{ac})

$$d_{ac}(v, B(K)) = \sum_{(i,j)\in K} (b_{ij} - \frac{v_i}{v_i + v_j})^2 \rightarrow \min;$$

s.t.

$$v_i \geq \varepsilon > 0, i = 1,2,\ldots,n, \sum_{j=1}^{n} v_j = 1$$

a-consistency based on deviations 2

Given $\boldsymbol{B}(K)$, i.e. $0 \leq b_{ij} \leq 1$, $(i,j) \in K \subseteq I^2$

$v^* = (v_1^*, \ldots, v_n^*)$ - optimal solution of $(\mathrm{P_{ac}})$

Define the nxn matrix $\boldsymbol{B}^{ac}(K)$ as follows:

$\boldsymbol{B}^{ac}(K) = \{c_{ij}\}$ where $c_{ij} = b_{ij}$ for all $(i,j) \in K$,

$$c_{ij} = \frac{v_i^*}{v_i^* + v_j^*} \qquad \text{for all } (i,j) \in I^2 - K$$

$\boldsymbol{B}^{ac}(K)$ is called **ac-extension matrix of** $B(K)$

Theorem 5. $\boldsymbol{B} = \boldsymbol{B}^{ac}(K)$, $0 \leq b_{ij} \leq 1$, $(i,j) \in K = I^2$

$\boldsymbol{B}$ is a-consistent iff $d_{ac}(v^*, B(K)) = 0$

a-transitivity based on deviations 1

$\boldsymbol{B}(K)$ - **fuzzy preference** matrix with missing elements with respect to $K = L \cup L' \cup D$, $|L| \geq n$-1

$\boldsymbol{u^*} = (u^*_1, u^*_2, \ldots, u^*_n)$ - **optimal solution** of optimization problem:

(P_{at})

$$d_{at}(u, B(K)) = \sum_{(i,j)\in K} \left(b_{ij} - \tfrac{1}{2}(1 + nu_i - nu_j)\right)^2 \to \min;$$

s.t.

$$u_i \geq \varepsilon > 0, i = 1, 2, \ldots, n, \sum_{j=1}^{n} u_j = 1$$

a-transitivity based on deviations 2

Given $\boldsymbol{B}(K)$, i.e. $0 \leq b_{ij} \leq 1$, $(i,j) \in K \subseteq I^2$

$u^* = (u_1^*, \ldots, u_n^*)$ be optimal solution of $(\mathrm{P}_{\mathrm{at}})$

Define the nxn matrix $\boldsymbol{B}^{at}(K)$ as follows:

$\boldsymbol{B}^{at}(K) = \{c_{ij}\}$ where $c_{ij} = b_{ij}$ for all $(i,j) \in K$,

$$c_{ij} = \max\{0, \min[1, \frac{1}{2}(1 + nu_i^* - nu_j^*)]\}$$ for all $(i,j) \in I^2 - K$

$\boldsymbol{B}^{at}(K)$ is called **at-extension matrix of** $B(K)$

Theorem 6. Let $\boldsymbol{B} = \boldsymbol{B}^{at}(K)$, $0 < b_{ij} < 1$, $(i,j) \in K = I^2$

If $\boldsymbol{B}$ is a-transitive then $d_{at}(u^*, B(K)) = 0$

Fuzzy preference matrix with missing elements: 2 special cases

Case 1: $L = \{(1,2), (2,3), \ldots, (n\text{-}1, n)\}$, $K = L \cup L' \cup D$

Example:

$$B(K) = \begin{pmatrix} 0.5 & 0.4 & - & - \\ 0.6 & 0.5 & 0.6 & - \\ - & 0.4 & 0.5 & 0.7 \\ - & - & 0.3 & 0.5 \end{pmatrix}$$

Case 2: $L = \{(1,2), (1,3), \ldots, (1, n)\}$, $K = L \cup L' \cup D$

Example:

$$B(K) = \begin{pmatrix} 0.5 & 0.9 & 0.8 & 0.3 \\ 0.1 & 0.5 & - & - \\ 0.2 & - & 0.5 & - \\ 0.7 & - & - & 0.5 \end{pmatrix}$$

Fuzzy preference matrix with missing elements 1: Case 1

Theorem 7. Given $\boldsymbol{B}(K)$, $L = \{(1,2), (2,3), \ldots, (n\text{-}1, n)\}$,
$K = L \cup L' \cup D$, $b_{ij} > 0$, $(i,j) \in L$

Then $\boldsymbol{B}^{ac}(K) = \{c_{ij}\}$ is a-consistent, where

$$c_{ij} = \frac{v_i^*}{v_i^* + v_j^*} \quad \text{for all } (i,j),$$

$v^* = (v_1^*, \ldots, v_n^*)$ can be explicitly calculated as follows:

$$\boxed{\begin{gathered} v_1^* = \frac{1}{S}, \qquad v_{i+1}^* = a_{i,i+1} v_1^*, \quad i = 1,2,..,n\text{-}1, \\ S = 1 + \sum_{i=1}^{n-1} a_{i,i+1} a_{i+1,i+2} \cdot \ldots \cdot a_{n-1,n}, \quad a_{ij} = \frac{1 - b_{ij}}{b_{ij}} \text{ for all } (i,j) \in K. \end{gathered}}$$

Fuzzy preference matrix with missing elements 2: Case 1

Theorem 8. Given $\boldsymbol{B}(K)$, $L = \{(1,2), (2,3), \ldots, (n\text{-}1,n)\}$, $K = L \cup L' \cup D$, $0 < b_{ij} < 1$, $(i,j) \in L$.

Then $\boldsymbol{B}^{at}(K) = \{c_{ij}\}$

where $c_{ij} = \max\{0, \min[1, \frac{1}{2}(1 + nu_i^* - nu_j^*)]\}$ for all (i,j)

and $u^* = (u_1^*, \ldots, u_n^*)$ can be iteratively calculated as:

$$u_i^* = \frac{2}{n^2}\sum_{j=1}^{n-1}\alpha_j - \frac{2}{n}\alpha_{i-1} - \frac{n-i-1}{n} \quad \text{for } i=1,2,\ldots,n,$$

where $\alpha_0 = 0, \alpha_j = \sum_{i=1}^{j} b_{i,i+1}$ for $j=1,2,\ldots,n\text{-}1$.

Moreover, $\boldsymbol{B}^{at}(K)$ is a-transitive iff

$$\left|\sum_{k=i}^{j-1} b_{k,k+1} - \frac{j-i}{2}\right| \le \frac{1}{2} \quad \text{for } i=1,2,\ldots,n\text{-}1,\ j=i+1,\ldots,n.$$

Example 6: a-reciprocal matrix with missing elements: Case 1

$$B(K)=\begin{pmatrix} 0.5 & 0.4 & - & - \\ 0.6 & 0.5 & 0.6 & - \\ - & 0.4 & 0.5 & 0.7 \\ - & - & 0.3 & 0.5 \end{pmatrix}$$

Theorem 7.

$$v_{i+1}^{*}=a_{i,i+1}v_{1}^{*},$$

$$a_{ij}=\frac{1-b_{ij}}{b_{ij}}$$

$$c_{ij}=\frac{v_i^{*}}{v_i^{*}+v_j^{*}}$$

$$B^{ac}(K)=\begin{pmatrix} 0.5 & 0.4 & \mathbf{0.5} & \mathbf{0.583} \\ 0.6 & 0.5 & 0.6 & \mathbf{0{,}677} \\ \mathbf{0.5} & 0.4 & 0.5 & 0.5 \\ \mathbf{0.417} & \mathbf{0.323} & 0.5 & 0.5 \end{pmatrix} \qquad \mathbf{v}=\begin{pmatrix} 0{,}237 \\ 0{,}356 \\ 0{,}237 \\ 0{,}169 \end{pmatrix}$$

Example 7: a-reciprocal matrix ac-extension: Case 1

$$B(K)=\begin{pmatrix} 0.5 & 0.6 & - & - & - & - & - \\ 0.4 & 0.5 & 0.5 & - & - & - & - \\ - & 0.5 & 0.5 & 0.4 & - & - & - \\ - & - & 0.6 & 0.5 & 0.3 & - & - \\ - & - & - & 0.7 & 0.5 & 0.2 & - \\ - & - & - & - & 0.8 & 0.5 & 0.1 \\ - & - & - & - & - & 0.9 & 0.5 \end{pmatrix}$$

Theorem 7.

$$c_{ij}=\frac{v_i^*}{v_i^*+v_j^*}$$

$$B^{ac}(K)=\begin{pmatrix} 0.5 & 0.6 & 0.600 & 0.500 & 0.300 & 0.097 & 0.012 \\ 0.4 & 0.5 & 0.5 & 0.400 & 0.222 & 0.067 & 0.008 \\ 0.400 & 0.5 & 0.5 & 0.4 & 0.222 & 0.067 & 0.008 \\ 0.500 & 0.600 & 0.6 & 0.5 & 0.3 & 0.097 & 0.012 \\ 0.700 & 0.778 & 0.778 & 0.7 & 0.5 & 0.2 & 0.027 \\ 0.903 & 0.933 & 0.933 & 0.903 & 0.8 & 0.5 & 0.1 \\ 0.988 & 0.992 & 0.992 & 0.988 & 0.973 & 0.9 & 0.5 \end{pmatrix}$$

$$\mathbf{v}=\begin{pmatrix} 0.010 \\ 0.007 \\ 0.007 \\ 0.010 \\ 0.024 \\ 0.094 \\ 0.848 \end{pmatrix}$$

Example 7: a-reciprocal matrix at-extension: Case 1

$$B(K)=\begin{pmatrix} 0.5 & 0.6 & - & - & - & - & - \\ 0.4 & 0.5 & 0.5 & - & - & - & - \\ - & 0.5 & 0.5 & 0.4 & - & - & - \\ - & - & 0.6 & 0.5 & 0.3 & - & - \\ - & - & - & 0.7 & 0.5 & 0.2 & - \\ - & - & - & - & 0.8 & 0.5 & 0.1 \\ - & - & - & - & - & 0.9 & 0.5 \end{pmatrix}$$

Theorem 8.

$$B^{at}(K)=\begin{pmatrix} 0.5 & 0.600 & 0.600 & 0.500 & 0.300 & 0.000 & 0.000 \\ 0.400 & 0.5 & 0.500 & 0.400 & 0.200 & 0.000 & 0.000 \\ 0.400 & 0.500 & 0.5 & 0.400 & 0.200 & 0.000 & 0.000 \\ 0.500 & 0.600 & 0.6 & 0.5 & 0.3 & 0.000 & 0.000 \\ 0.700 & 0.800 & 0.800 & 0.700 & 0.5 & 0.200 & 0.000 \\ 1.000 & 1.000 & 1.000 & 1.000 & 0.800 & 0.5 & 0.100 \\ 1.000 & 1.000 & 1.000 & 1.000 & 1.000 & 0.900 & 0.5 \end{pmatrix} \qquad \mathbf{v}=\begin{pmatrix} 0.086 \\ 0.057 \\ 0.057 \\ 0.086 \\ 0.143 \\ 0.229 \\ 0.343 \end{pmatrix}$$

This matrix is not a-transitive ! $\left|\sum_{k=i}^{j-1} b_{k,k+1} - \frac{j-i}{2}\right| \le \frac{1}{2}$ for $i=1,2,\ldots,n-1$, $j=i+1,\ldots,n$ is not satisfied!

Fuzzy preference matrix with missing elements 1: Case 2

Theorem 9. Given $\boldsymbol{B}(K)$, $L=\{(1,2),(1,3),\ldots,(1,n)\}$, $K=L\cup L'\cup D$.

Then $\boldsymbol{B}^{ac}(K)=\{c_{ij}\}$ is a-consistent, and c_{ij} can be explicitly calculated as follows:

$$c_{ij}=\frac{v_i^*}{v_i^*+v_j^*} \quad \text{for all } (i,j)\in K,$$

$$\boxed{\begin{array}{l} v_1^*=\dfrac{1}{V}, \qquad v_{i+1}^*=a_{1,i+1}v_1^*, \qquad i=1,2,..,n\text{-}1, \\[2ex] V=1+\displaystyle\sum_{i=1}^{n-1}a_{1,i+1}, \quad a_{ij}=\dfrac{1-b_{ij}}{b_{ij}} \quad \text{for all } (i,j)\in K. \end{array}}$$

Fuzzy preference matrix with missing elements 2: Case 2

Theorem 10. Given $\boldsymbol{B}(K)$, $L = \{(1,2), (1,3),\ldots,(1,n)\}$, $K = L \cup L' \cup D$.

Then $\boldsymbol{B}^{ac}(K)=\{c_{ij}\}$ can be explicitly calculated as follows:

$$c_{ij} = \max\{0, \min[1, \frac{1}{2}(1 + nu_i^* - nu_j^*)]\} \quad \text{for all } (i,j) \in K,$$

$$\boxed{\begin{aligned} u_1^* &= \frac{2}{n^2}\sum_{j=1}^{n-1} b_{1,j+1} \\ u_{i+1}^* &= u_1^* + \frac{1-2b_{1,i+1}}{n} \quad \text{for } i=1,2,\ldots,n-1. \end{aligned}}$$

and $\boldsymbol{B}^{at}(K)$ is **a-transitive** iff

$$\boxed{\left|b_{1j} - b_{1i}\right| \le \frac{1}{2} \quad \text{for all } i,j}$$

Example 8: a-reciprocal matrix ac-extension: Case 2

$$B(K)=\begin{pmatrix} 0.5 & 0.9 & 0.8 & 0.3 \\ 0.1 & 0.5 & - & - \\ 0.2 & - & 0.5 & - \\ 0.7 & - & - & 0.5 \end{pmatrix}$$

Theorem 9.

$$c_{ij}=\frac{v_i^*}{v_i^*+v_j^*}$$

$$\mathbf{v}=\begin{pmatrix} 0{,}237 \\ 0{,}356 \\ 0{,}237 \\ 0{,}169 \end{pmatrix}$$

$$B^{ac}(K)=\begin{pmatrix} 0.5 & 0.9 & 0.8 & 0.3 \\ 0.1 & 0.5 & \mathbf{0.03} & \mathbf{0.04} \\ 0.2 & \mathbf{0.97} & 0.5 & \mathbf{0.10} \\ 0.7 & \mathbf{0.94} & \mathbf{0.90} & 0.5 \end{pmatrix}$$

Example 9: a-reciprocal matrix ac-extension: Case 2

$$B(K)=\begin{pmatrix} 0.5 & 0.6 & 0.5 & 0.4 & 0.3 & 0.2 & 0.1 \\ 0.4 & 0.5 & - & - & - & - & - \\ 0.5 & - & 0.5 & - & - & - & - \\ 0.6 & - & - & 0.5 & - & - & - \\ 0.7 & - & - & - & 0.5 & - & - \\ 0.8 & - & - & - & - & 0.5 & - \\ 0.9 & - & - & - & - & - & 0.5 \end{pmatrix}$$

Theorem 9.

$$\mathbf{v}=\begin{pmatrix} 0.051 \\ 0.034 \\ 0.051 \\ 0.077 \\ 0.120 \\ 0.205 \\ 0.462 \end{pmatrix}$$

$$c_{ij}=\frac{v_i^*}{v_i^*+v_j^*}$$

$$B^{ac}(K)=\begin{pmatrix} 0.5 & 0.600 & 0.500 & 0.400 & 0.300 & 0.200 & 0.100 \\ 0.400 & 0.5 & 0.400 & 0.308 & 0.222 & 0.143 & 0.069 \\ 0.500 & 0.600 & 0.5 & 0.400 & 0.300 & 0.200 & 0.100 \\ 0.600 & 0.692 & 0.600 & 0.5 & 0.391 & 0.273 & 0.143 \\ 0.700 & 0.778 & 0.700 & 0.609 & 0.5 & 0.368 & 0.206 \\ 0.800 & 0.857 & 0.800 & 0.727 & 0.632 & 0.5 & 0.308 \\ 0.900 & 0.931 & 0.900 & 0.857 & 0.794 & 0.692 & 0.5 \end{pmatrix}$$

Example 10: a-reciprocal matrix at-extension: Case 2

$$B(K)=\begin{pmatrix} 0.5 & 0.6 & 0.5 & 0.4 & 0.3 & 0.2 & 0.1 \\ 0.4 & 0.5 & - & - & - & - & - \\ 0.5 & - & 0.5 & - & - & - & - \\ 0.6 & - & - & 0.5 & - & - & - \\ 0.7 & - & - & - & 0.5 & - & - \\ 0.8 & - & - & - & - & 0.5 & - \\ 0.9 & - & - & - & - & - & 0.5 \end{pmatrix}$$

Theorem 10.

$$\mathbf{v}=\begin{pmatrix} 0.106 \\ 0.078 \\ 0.106 \\ 0.135 \\ 0.163 \\ 0.193 \\ 0.222 \end{pmatrix}$$

$$B^{at}(K)=\begin{pmatrix} 0.5 & 0.600 & 0.500 & 0.400 & 0.300 & 0.200 & 0.100 \\ 0.400 & 0.5 & 0.400 & 0.300 & 0.200 & 0.100 & 0.000 \\ 0.500 & 0.600 & 0.5 & 0.400 & 0.300 & 0.200 & 0.100 \\ 0.600 & 0.700 & 0.600 & 0.5 & 0.400 & 0.300 & 0.200 \\ 0.700 & 0.800 & 0.700 & 0.600 & 0.5 & 0.400 & 0.300 \\ 0.800 & 0.900 & 0.800 & 0.700 & 0.600 & 0.5 & 0.400 \\ 0.900 & 1.000 & 0.900 & 0.800 & 0.700 & 0.600 & 0.5 \end{pmatrix}$$

$$\left|b_{1j}-b_{1i}\right|\le\frac{1}{2} \qquad \forall i,j=1,2,\ldots,n.$$

$$c_{ij}=\max\{0,\min[1,\frac{1}{2}(1+nu_i^*-nu_j^*)]\}$$

Conclusions

- We defined two types of pair-wise comparison relations (**multiplicative** and **additive**) as well as the corresponding concepts of **reciprocity** and **consistency/transitivity.**
- We derived mutual relationships between two types of consistency: multiplicative and additive consistency for measuring a-consistency/a-transitivity of matrices.
- We investigated two alternative ways for dealing with fuzzy decision matrices concerning compatibility of pairwise comparisons of elements: **a-consistency** and **a-transitivity.**
- We designed an approach based on the LSQ method which allows for dealing with missing elements, e.g. large matrices.
- Two specific cases of matrices with missing elements have been investigated and demonstrated on examples.

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THANK YOU!