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On the Kluvánek construction of the Lebesgue integral

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$\mathcal{C}$ - functions continuous on $[a, b]$

$\mathcal{R}$ - Riemann integrable functions

$\mathcal{L}$ - Lebesgue integrable functions

$\mathcal{C} \subset \mathcal{R} \subset \mathcal{L}$

$$d(f,g) = \int_a^b |\, f(x) - g(x) \,|\, dx$$

$(\mathcal{L}, d)$ is a complete pseudometric space

Problem: How to define the Lebesgue integral without the measure theory

$\mathcal{A}$ - the family of all subintervals of $[a, b]$

$$\mu([c, d]) = d - c$$

Kluvánek construction

$$\exists\, \alpha_i \in R, \exists A_i \in \mathcal{A},\ \sum_{i=1}^{\infty} |\alpha_i| \mu(A_i) < \infty$$

$$\sum |\alpha_i| \chi_{A_i}(x) < \infty \Longrightarrow f(x) = \sum \alpha_i \chi_{A_i}(x)$$

$$\int_a^b f(x) dx = \sum_{i=1}^{\infty} \alpha_i \mu(A_i)$$

Problem

If $f = \sum_{i=1}^{n} \alpha_i \chi_{A_i} = \sum_{j=1}^{m} \beta_j \chi_{B_j}$

then it is necessary to prove

$$\sum_{i=1}^{n} \alpha_i \mu(A_i) = \sum_{j=1}^{m} \beta_j \mu(B_j).$$

Kluvánek, I.: Archimedes was right. Elemente der Mathematik 42 (1987), No 3, 51 - 82, No 4, 83-114.

Kluvánek, I.: Integral Calculus (in Slovak). Ružomberok UK 2011.

The Kluvánek proof depends on some properties of the real line.

A modification of the Kluvánek definition:

first for non - negative functions only

A nonnegative function f is integrable ($f \in \mathcal{P}^+$), if

$$\exists\, \alpha_i \geq 0, \exists A_i \in \mathcal{A},\ \sum_{i=1}^{\infty} \alpha_i \mu(A_i) < \infty$$

$$\sum \alpha_i \chi_{A_i}(x) < \infty \Longrightarrow f(x) = \sum \alpha_i \chi_{A_i}(x)$$

Theorem 1. $f : [a,b] \to [-\infty, \infty]$ is integrable in the Kluvánek sense if and only if there exist $g, h \in \mathcal{P}^+$, such that

$$g(x) < \infty, h(x) < \infty \Longrightarrow f(x) = g(x) - h(x).$$

Theorem 2. Let $f \in \mathcal{P}^+$

$$f = \sum_i \alpha_i \chi_{A_i} = \sum_j \beta_j \chi_{B_j}.$$

Then

$$\sum_i \alpha_i \mu(A_i) = \sum_i \beta_j \mu(B_j).$$

Definition 1. If

$$f = \sum_i \alpha_i \chi_{A_i}$$

then

$$\int_a^b f(x)dx = \sum_i \alpha_i \mu(A_i).$$

Definition 2. If there exists $g, h \in \mathcal{P}^+$, such that

$$g(x) < \infty, h(x) < \infty \Longrightarrow f(x) = g(x) - h(x)$$

then

$$\int_a^b f(x)dx = \int_a^b g(x)dx - \int_a^b h(x)dx.$$

General formulation

X - non-empty set
$\mathcal{A}$ - ring of subsets of X
$\mathcal{A} \neq \emptyset$
$A, B \in \mathcal{A} \Longrightarrow A \cup B \in \mathcal{A}, A \setminus B \in \mathcal{A}$

$(X, \mathcal{A})$

$\mu : \mathcal{A} \to [0, \infty)$, σ-additive

$$A \in \mathcal{A}, A = \bigcup_{i=1}^{\infty} A_i, A_i \in \mathcal{A}(i = 1, 2, ...), A_i \cap A_j = \emptyset(i \neq j)$$

$$\Longrightarrow \mu(A) = \sum_{i=1}^{\infty} \mu(A_i)$$

Definition 3. A nonnegative function $f : X \to [0, \infty]$ is integrable ($f \in \mathcal{P}^+$), if

$$\exists\, \alpha_i \geq 0, \exists A_i \in \mathcal{A},\ \sum_{i=1}^{\infty} \alpha_i \mu(A_i) < \infty$$

$$\sum \alpha_i \chi_{A_i}(x) < \infty \Longrightarrow f(x) = \sum \alpha_i \chi_{A_i}(x).$$

Theorem 3. Let $f \in \mathcal{P}^+$

$$f = \sum_i \alpha_i \chi_{A_i} = \sum_j \beta_j \chi_{B_j}.$$

Then

$$\sum_i \alpha_i \mu(A_i) = \sum_i \beta_j \mu(B_j).$$

Definition 4. If

$$f = \sum_i \alpha_i \chi_{A_i}$$

then

$$\int_X f \, d\mu = \sum_i \alpha_i \, \mu(A_i).$$

Definition 5.
A function $f : X \to [-\infty, \infty]$ is integrable if there exist $g, h \in \mathcal{P}^+$, such that

$$g(x) < \infty, h(x) < \infty \Longrightarrow f(x) = g(x) - h(x).$$

$$\int_X f d\mu = \int_X g d\mu - \int_X h d\mu$$

Theorem 4. Let $(f_n)_n$ be a nondecreasing sequence of integrable functions, $\lim_{n\to\infty} \int_X f_n \, d\mu < \infty$. Then $\lim_{n\to\infty} f_n$ is integrable, and

$$\int_X \lim_{n\to\infty} f_n \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu.$$