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MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY

OP Vzdělávání pro konkurenceschopnost

INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ

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Olomoucian Days of Applied Mathematics

ODAM 2013

Department of Mathematical analysis
and Applications of Mathematics
Faculty of Science
Palacký Univerzity Olomouc

BAYESIAN STATISTICS AND FUZZY INFORMATION

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www.statistik.tuwien.ac.at/public/viertl/

BAYESIAN INFERENCE

$X \sim f(\cdot|\theta)$, $\theta \in \Theta$, $\tilde{\theta}$ Stochastic Qu.

$\pi(\cdot)$ a-priori distribution on Θ

$x_1, \cdots, x_n$ Sample information

Updating of the a-priori distribution

$$\underbrace{\pi(\theta | x_1, \cdots, x_n)}_{\text{a-posteriori distribution}} = \frac{\pi(\theta) \cdot \ell(\theta; x_1, \cdots, x_n)}{\int_\Theta \pi(\theta) \cdot \ell(\theta; x_1, \cdots, x_n)\, d\theta} \quad \forall \theta \in \Theta$$

$$\ell(\theta; x_1, \cdots, x_n) = \prod_{i=1}^{n} f(x_i | \theta)$$

FUZZY INFORMATION

- Fuzzy Data
- Fuzzy a-priori Knowledge
- Fuzzy Probabilities
- Soft Computing ECSC

FUZZY DATA

- Environmental Data
- Recovering Times
- Quality of Life Data
- Migration Data

⋮

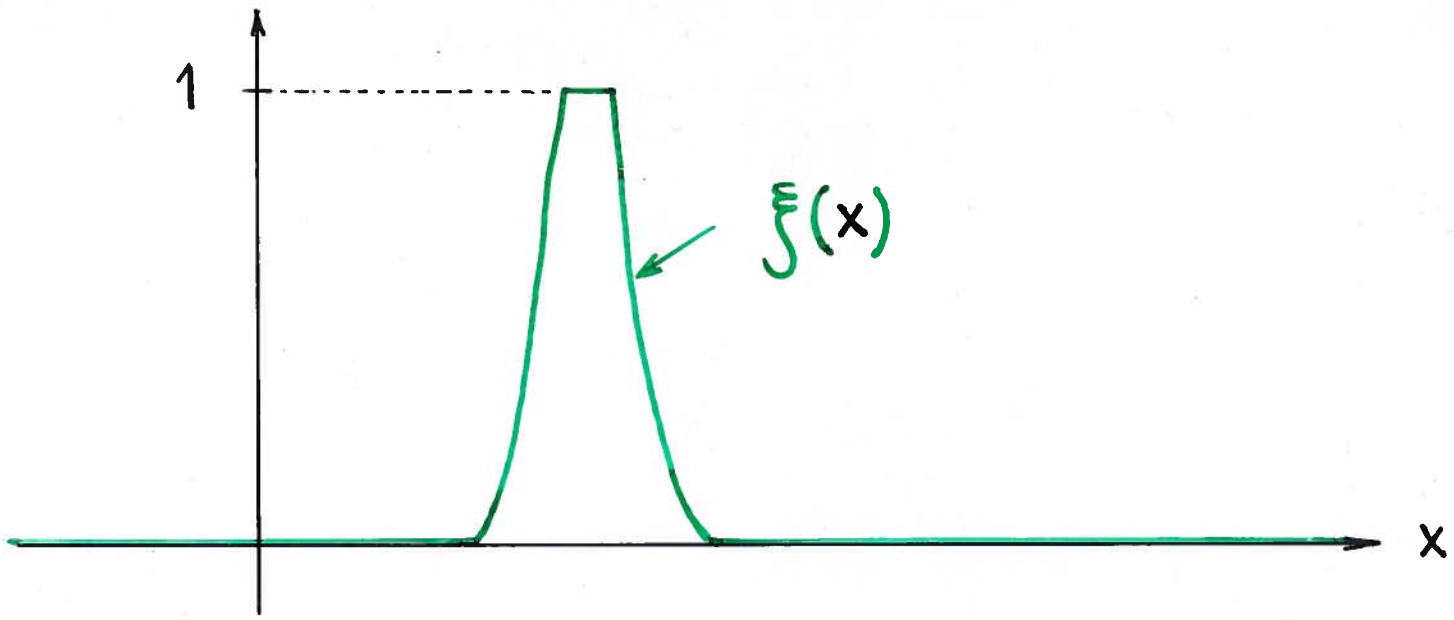
- Precision Measurement Data

MEASUREMENT RESULTS

Not precise numbers but more or less non-precise

Mathematical model : Fuzzy number x^*

Characterizing function $\xi(\cdot)$



Characterizing Function $\xi(\cdot)$

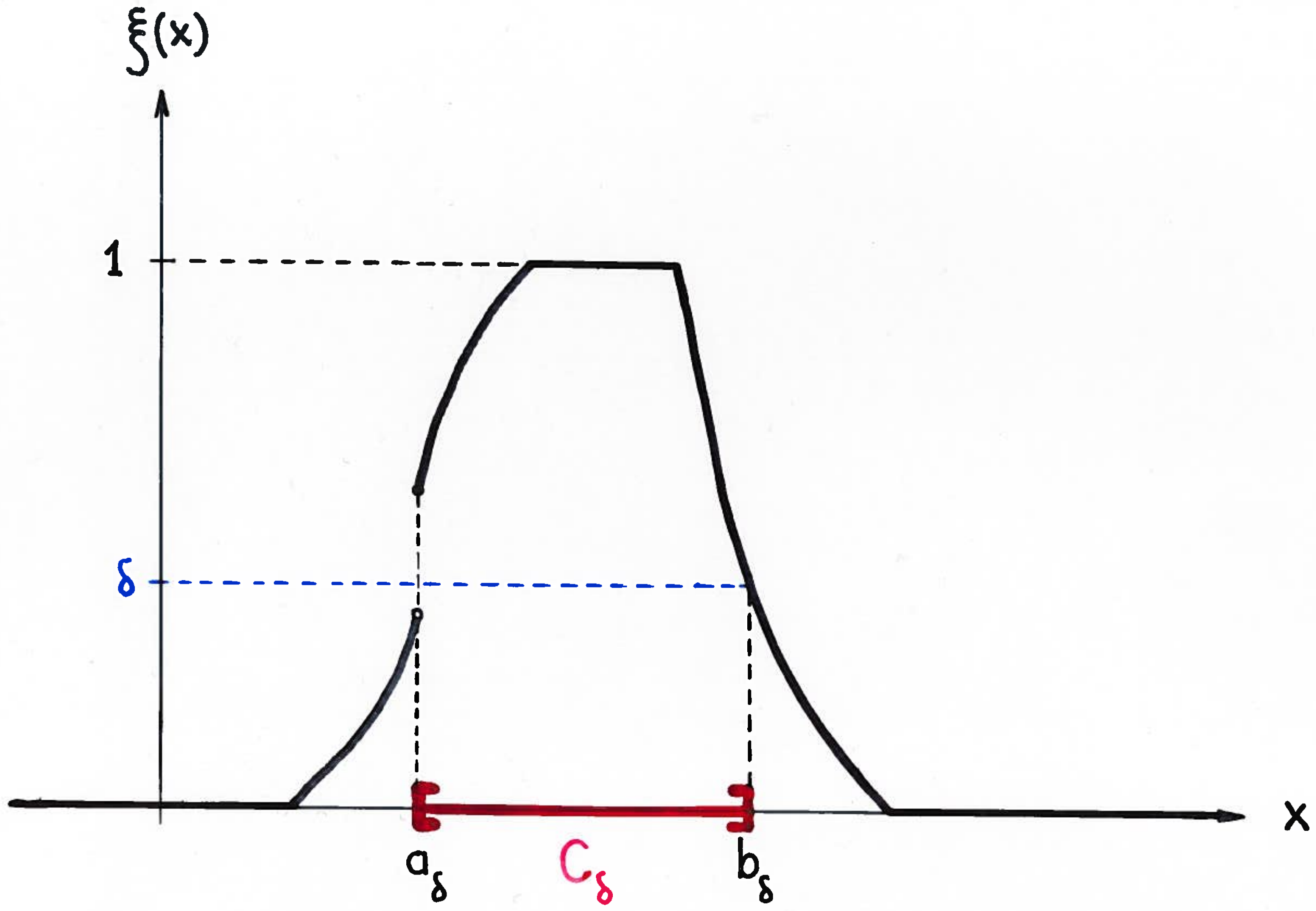
(1) $0 \leq \xi(x) \leq 1 \quad \forall x \in \mathbb{R}$

(2) support $[\xi(\cdot)]$ is bounded

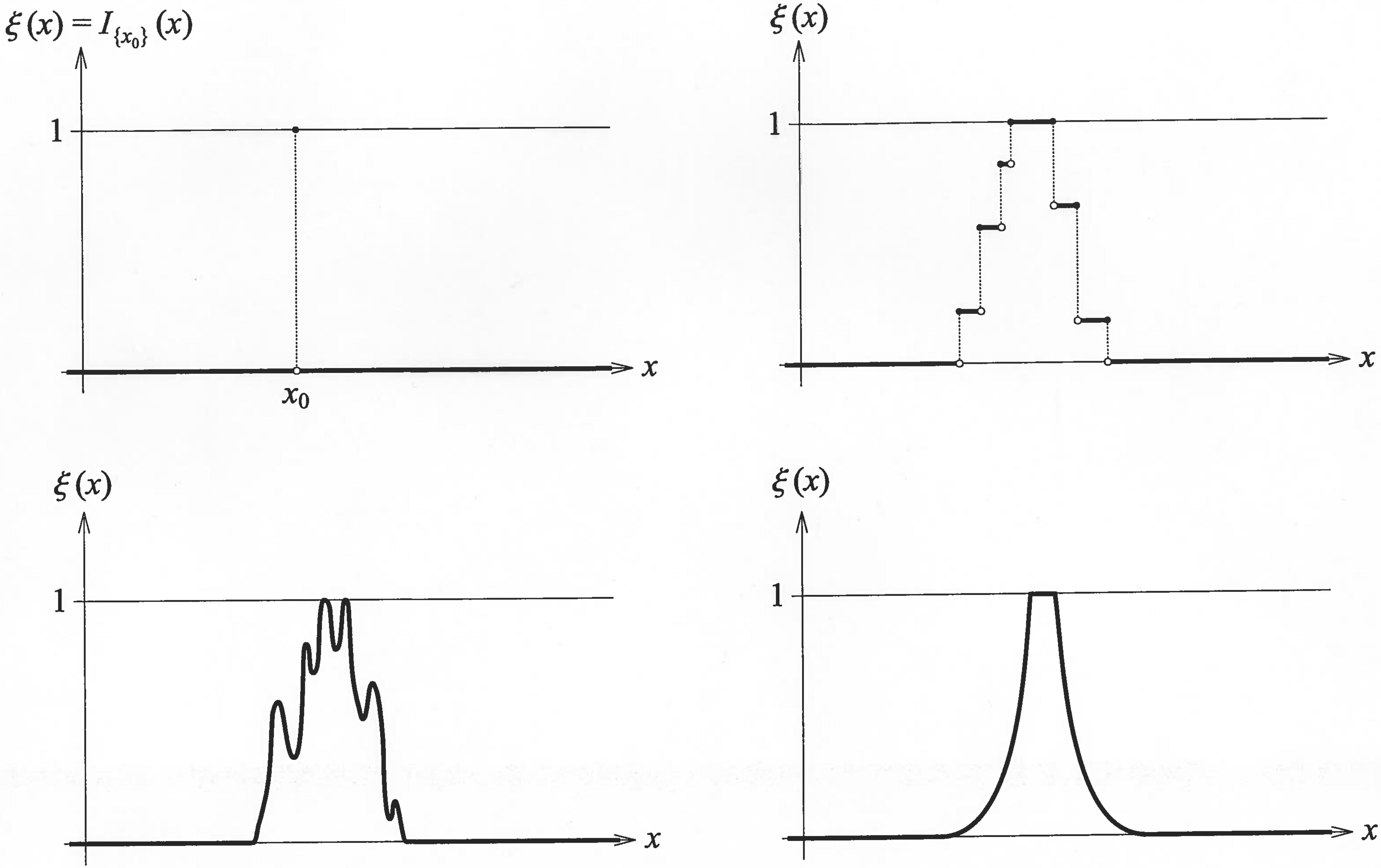
(3) $\forall \delta \in (0;1]$ the δ-Cut C_δ

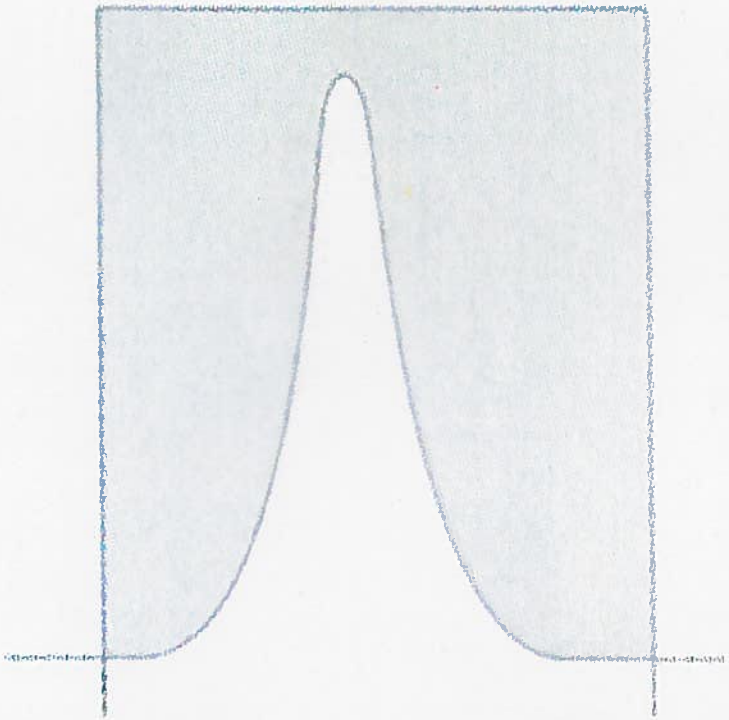
$$C_\delta = \{x \in \mathbb{R} : \xi(x) \geq \delta\} \neq \emptyset$$

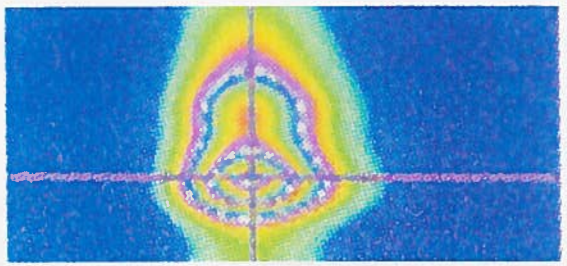
is a closed interval $[a_\delta ; b_\delta]$



ξ(x)
1
δ
x
a_δ
C_δ
b_δ


ξ(x) = I_{x_0}(x)
1
x
x_0
ξ(x)
1
x
ξ(x)
1
x
ξ(x)
1
x





Lemma: For a characterizing function of a fuzzy number the following holds:

$$\xi(x) = \max\left\{\delta \cdot \mathbb{1}_{C_\delta[\xi(\cdot)]}(x) : \delta \in [0;1]\right\} \quad \forall x \in \mathbb{R}$$

$\forall x \in R$

Remark: Not all families of nested closed intervals $[a_\delta; b_\delta]$ are the δ-cuts of a fuzzy number.

But the following definition yields a fuzzy number:

$$\xi(x) := \sup\left\{\delta \cdot \mathbb{1}_{[a_\delta; b_\delta]}(x) : \delta \in [0;1]\right\} \quad \forall x \in \mathbb{R}$$

NON-PRECISE NUMBERS

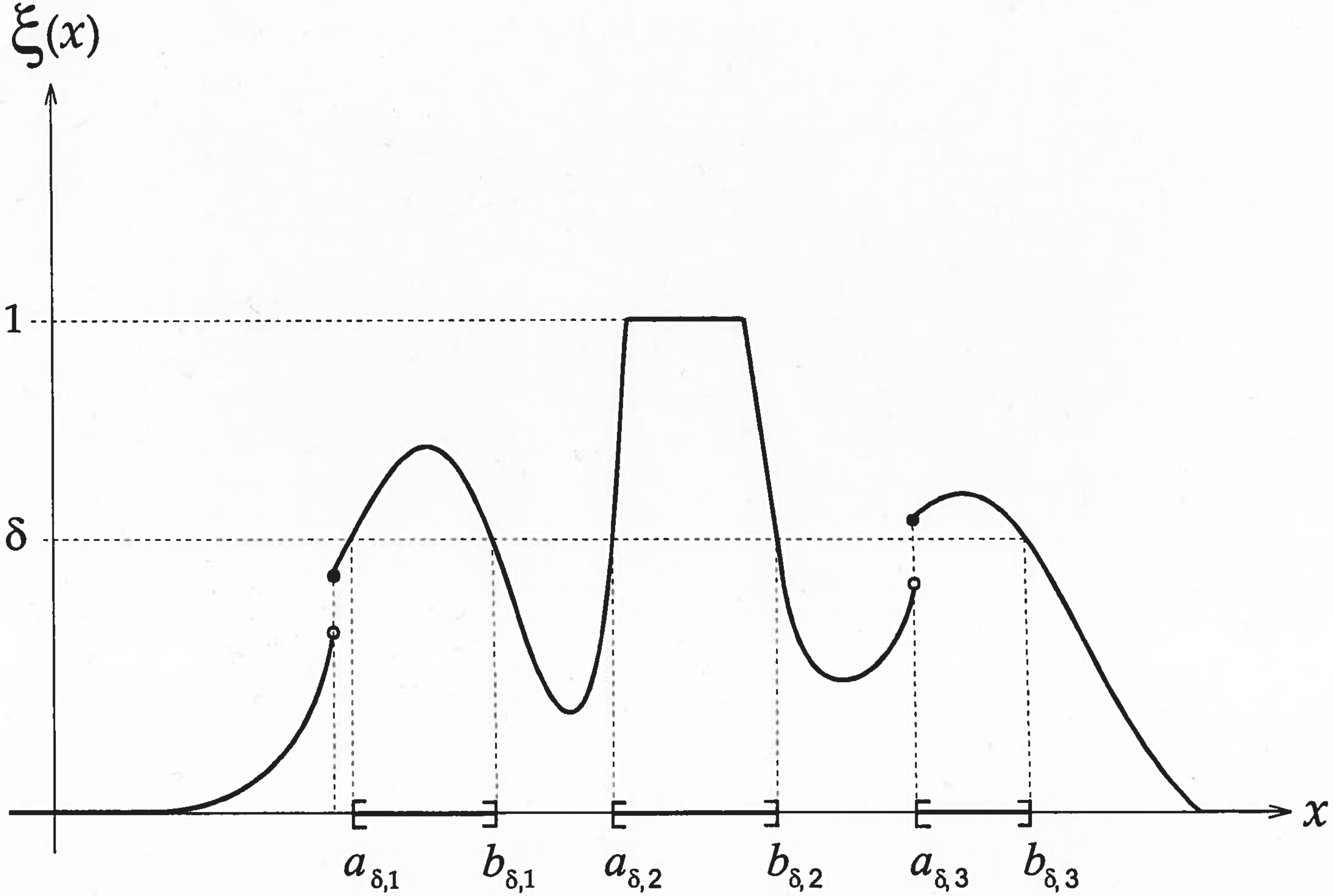
x^*, Characterizing Function $\xi(\cdot)$

(1) Support $[\xi(\cdot)] \subseteq [a; b]$ compact interval

(2) All δ-Cuts $C_\delta := \{x \in \mathbb{R} : \xi(x) \geq \delta\}$

are non-empty with

$$C_\delta = \bigcup_{j=1}^{k_\delta} [a_{\delta,j}; b_{\delta,j}], \quad k_\delta \in \mathbb{N}$$



ξ(x)
1
δ
x
$a_{\delta,1}$
$b_{\delta,1}$
$a_{\delta,2}$
$b_{\delta,2}$
$a_{\delta,3}$
$b_{\delta,3}$

FOR FUZZY DATA ?

Fuzzy valued functions $f^*: M \to \mathcal{F}_I(\mathbb{R})$

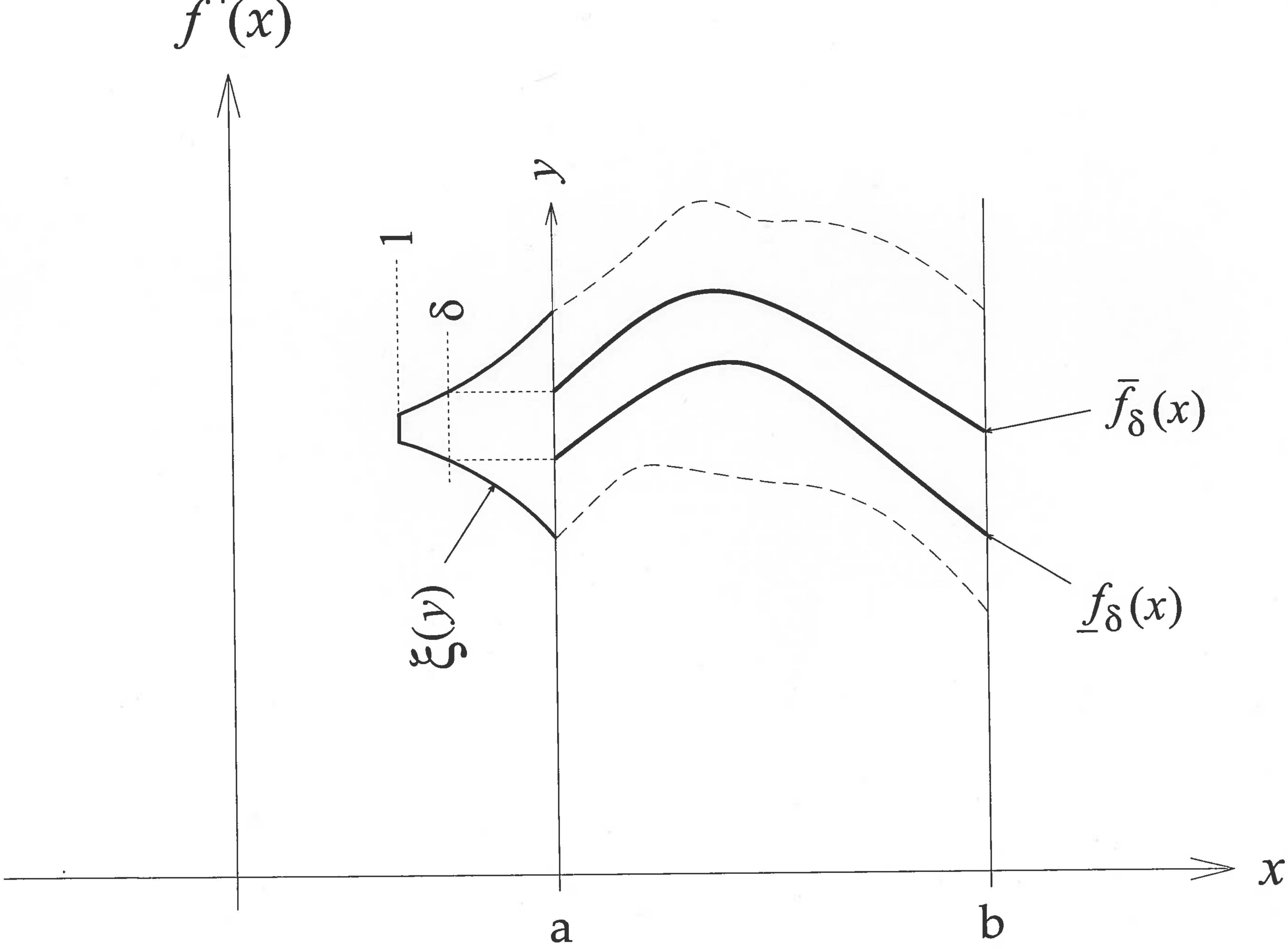
$$f^*(x) = y^* \mathrel{\widehat{=}} \xi_x(\cdot) \quad \forall x \in M$$

δ-level functions $\underline{f}_\delta(\cdot)$ and $\overline{f}_\delta(\cdot)$

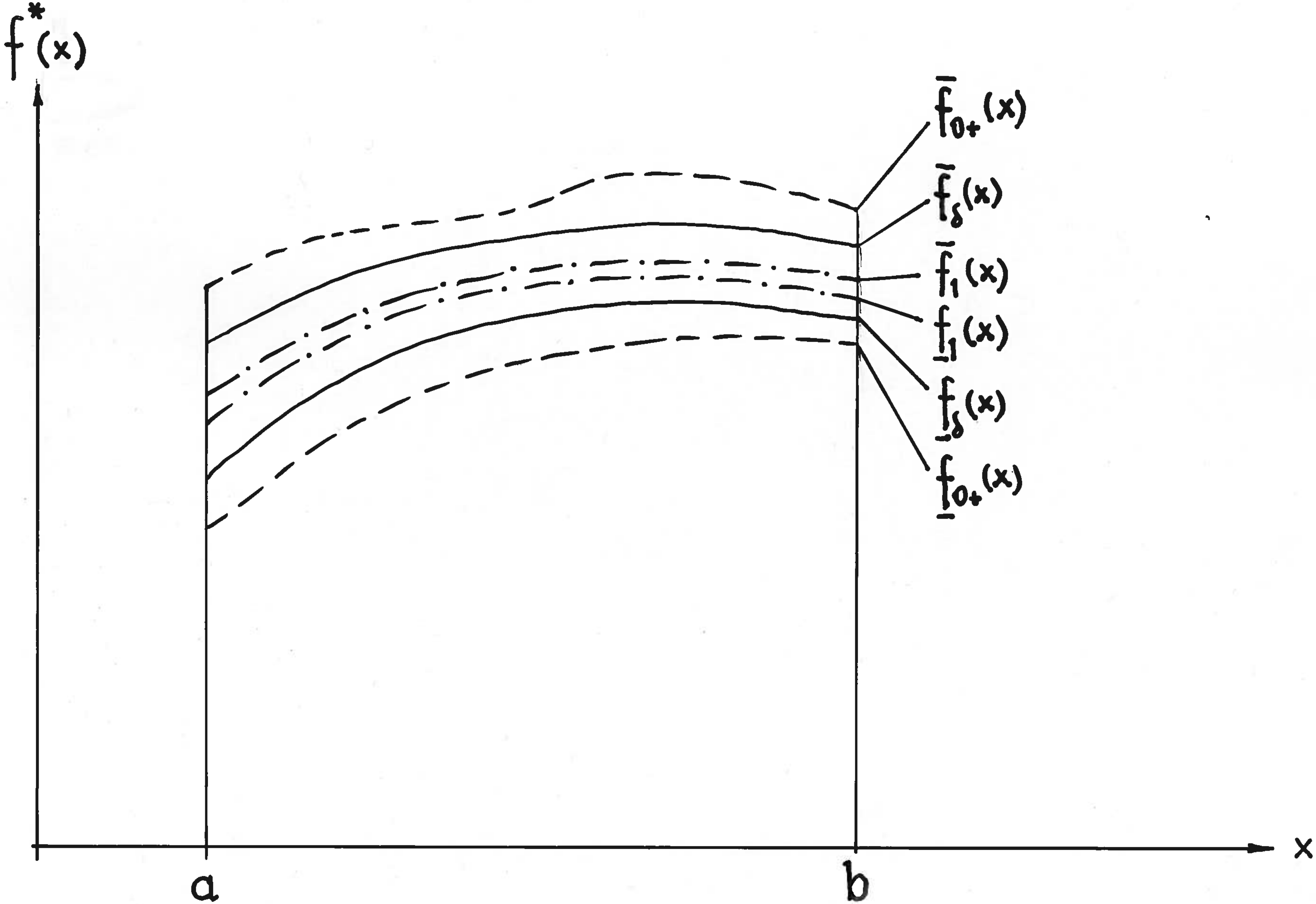
defined by $C_\delta[f^*(x)] = [\underline{f}_\delta(x), \overline{f}_\delta(x)] \quad \forall x \in M$

$\forall \delta \in (0; 1]$

For $M = \mathbb{R}$ δ-level curves (real functions)



$f^*(x)$
y
1
δ
$\xi(y)$
$\bar{f}_\delta(x)$
$\underline{f}_\delta(x)$
x
a
b


$f^*(x)$
$\bar{f}_{0+}(x)$
$\bar{f}_\delta(x)$
$\bar{f}_1(x)$
$\underline{f}_1(x)$
$\underline{f}_\delta(x)$
$\underline{f}_{0+}(x)$
x
a
b

FUZZY PROBABILITY DENSITY

Generalized densities $f^*(\cdot)$ on $\mathbb{R}$:

$f^*(\cdot)$ fuzzy function with δ-level functions $\underline{f}_\delta(\cdot)$ and $\overline{f}_\delta(\cdot)$ integrable with

$$\int_{\mathbb{R}} \overline{f}_\delta(x)\,dx < \infty \quad \forall \delta \in (0;1]$$

and $\exists$ classical density $f(\cdot)$ on $\mathbb{R}$ with

$$\underline{f}_1(x) \leq f(x) \leq \overline{f}_1(x) \quad \forall x \in \mathbb{R}$$

The fuzzy probability $P^*(B)$ of $B \in \mathcal{B}$ is a fuzzy interval

COMBINED FUZZY SAMPLE

Sample $x_1^*, \dots, x_n^*$

$\xi_1(\cdot), \dots, \xi_n(\cdot)$

x_i^* Fuzzy Element of Observation Space M

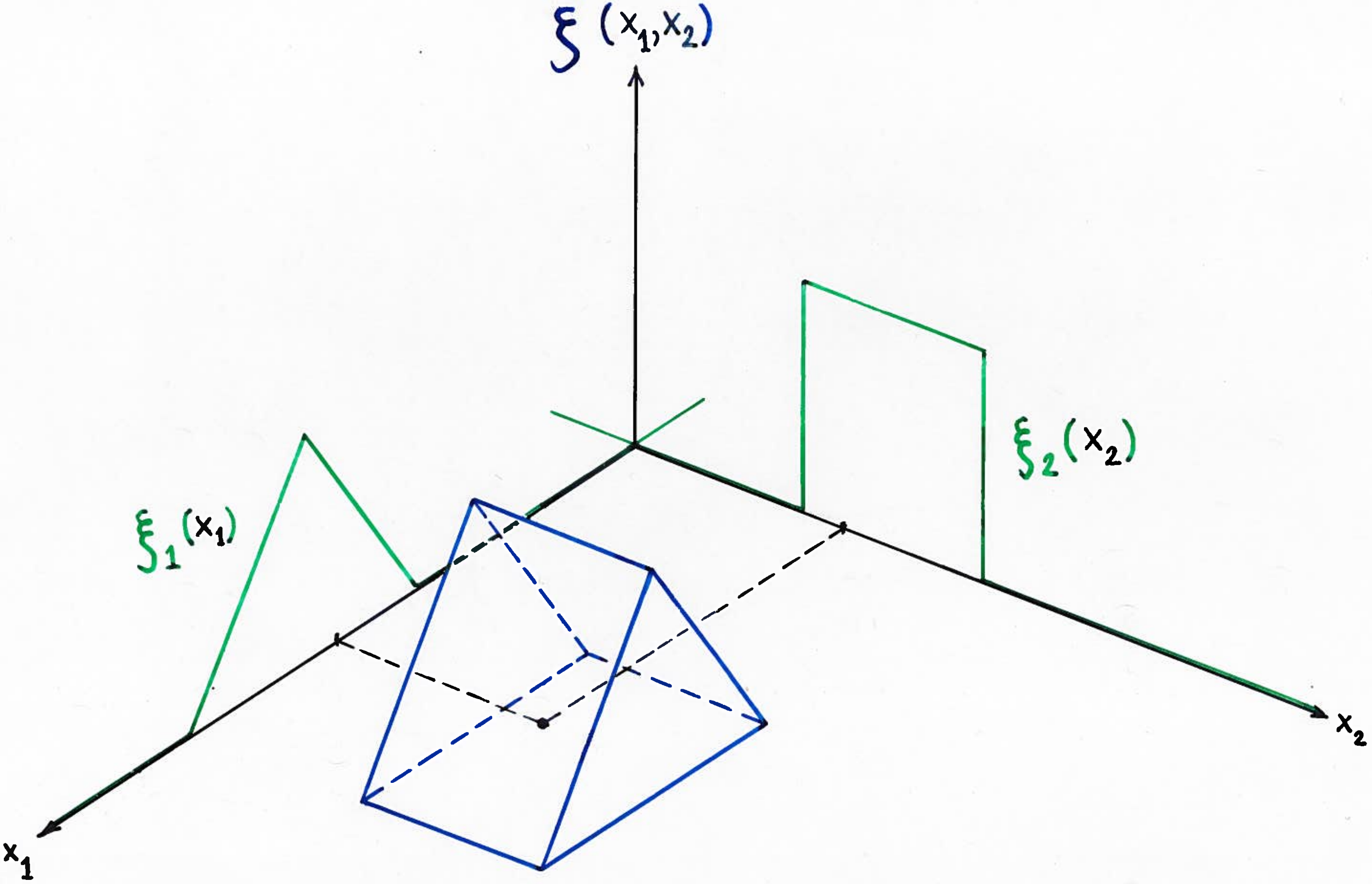
$M^n = \{ \underline{x} = (x_1, \dots, x_n) : x_i \in M \}$ Sample Space

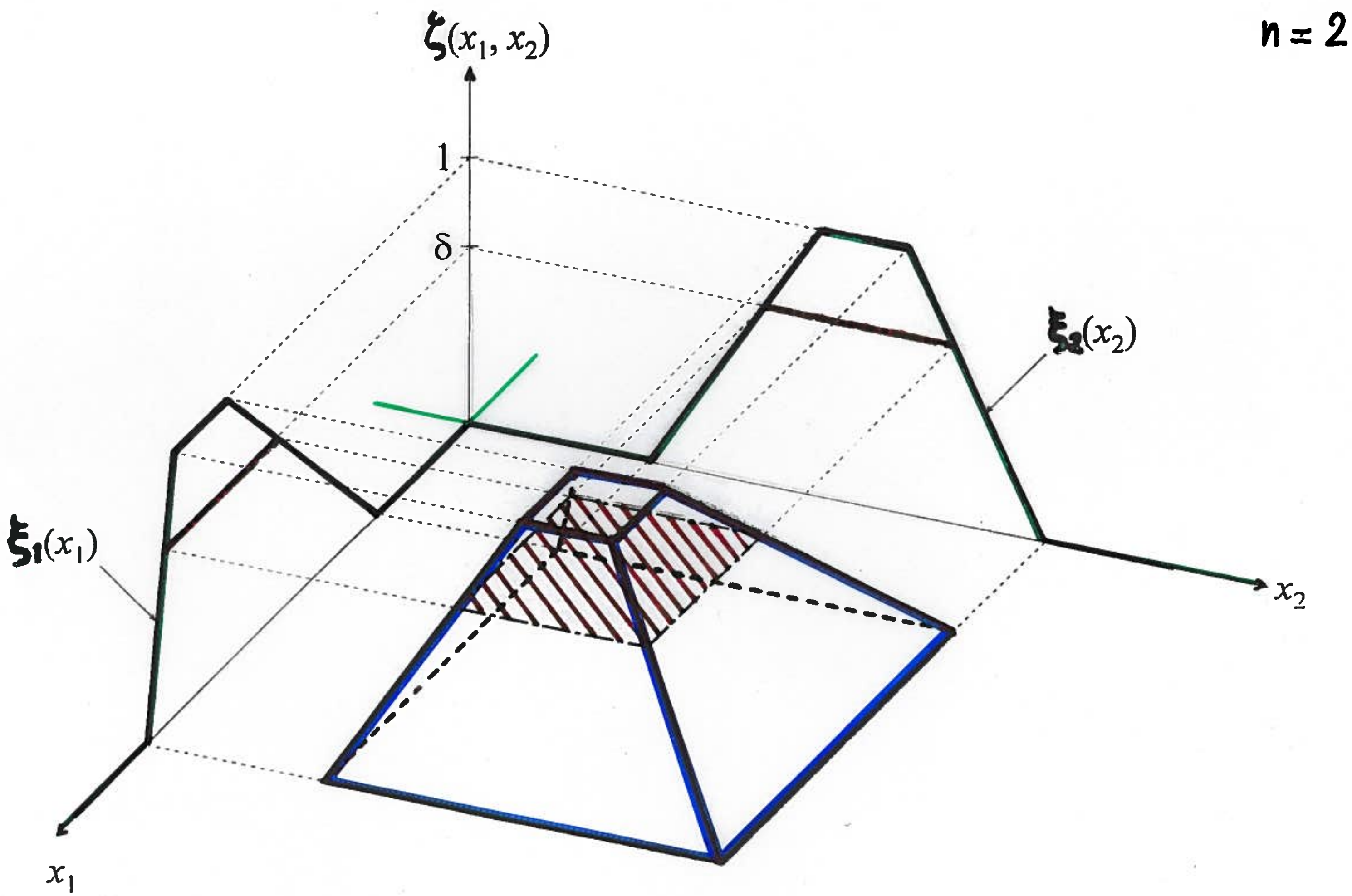
$\underline{x}^*$ Fuzzy Element of M^n with VCF $\xi(\cdot)$

$$\xi(x_1, \dots, x_n) = T_n\left(\xi_1(x_1), \dots, \xi_n(x_n)\right) \quad \forall (x_1, \dots, x_n)$$

$\underline{x}^*$ Combined Fuzzy Sample

$n = 2$





$$\xi(\underline{x}) := \min_{i=1(1)n} \xi_i(x_i)$$

PROBLEMS

Sequential updating

Precise a-priori density

ALTERNATIVE SOLUTION

Based on δ-level functions

$$\overline{\pi}_\delta(\cdot), \quad \overline{\ell}_\delta(\cdot\,;\underline{x}^*), \quad \overline{\pi}_\delta(\cdot\,|\,\underline{x}^*)$$

$$\underline{\pi}_\delta(\cdot), \quad \underline{\ell}_\delta(\cdot\,;\underline{x}^*), \quad \underline{\pi}_\delta(\cdot\,|\,\underline{x}^*)$$

LIKELIHOOD FOR FUZZY DATA

$\underline{x}^*$ combined fuzzy sample with v.c.f. $\xi(\cdot)$

$\ell^*(\theta;\underline{x}^*)$ fuzzy value of the likelihood $\ell(\theta;\underline{x})$ with c.f. $\eta_\theta(\cdot)$ defined by

$$\eta_\theta(y) = \left\{ \begin{array}{ll} \sup\{\xi(\underline{x}) : \ell(\theta;\underline{x}) = y\} & \text{if } \ell^{-1}(\{y\}) \neq \emptyset \\ 0 & \text{if } \ell^{-1}(\{y\}) = \emptyset \end{array} \right\} \quad \forall y \in \mathbb{R}$$

Remark: For precise data $\underline{x}$ the indicator function of $\ell(\theta;\underline{x})$ is obtained

GENERALIZED BAYES' THEOREM

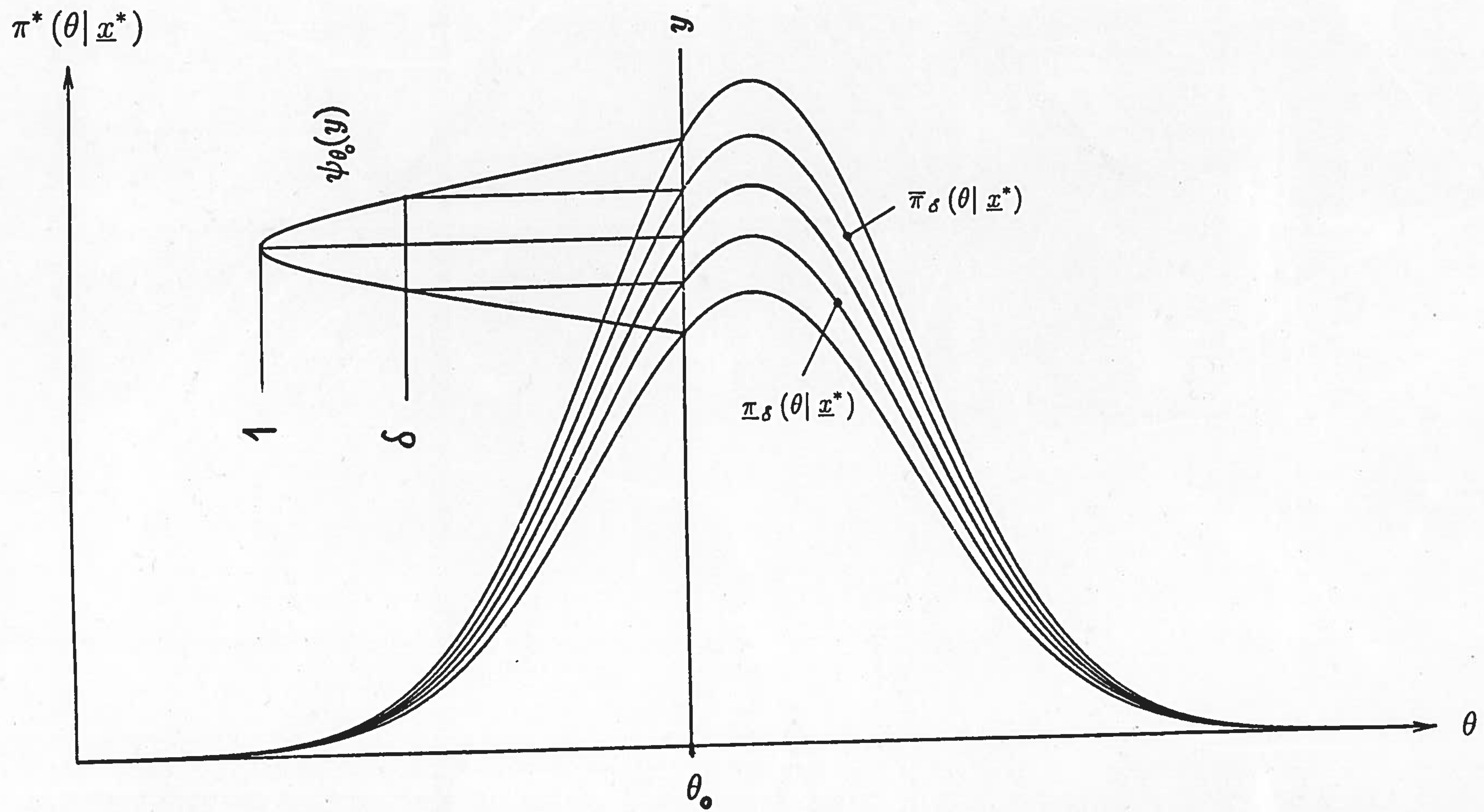
δ-level curves of the fuzzy a-posteriori density

$$\overline{\pi}_\delta(\theta|\underline{x}^*) = \frac{\overline{\pi}_\delta(\theta)\,\overline{\ell}_\delta(\theta;\underline{x}^*)}{\int_\Theta \frac{1}{2}\left[\underline{\pi}_\delta(\theta)\,\underline{\ell}_\delta(\theta;\underline{x}^*) + \overline{\pi}_\delta(\theta)\,\overline{\ell}_\delta(\theta;\underline{x}^*)\right] d\theta}$$

$$\underline{\pi}_\delta(\theta|\underline{x}^*) = \frac{\underline{\pi}_\delta(\theta)\,\underline{\ell}_\delta(\theta;\underline{x}^*)}{\int_\Theta \frac{1}{2}\left[\underline{\pi}_\delta(\theta)\,\underline{\ell}_\delta(\theta;\underline{x}^*) + \overline{\pi}_\delta(\theta)\,\overline{\ell}_\delta(\theta;\underline{x}^*)\right] d\theta}$$

$$\forall\, \theta \in \Theta$$

Figure *Fuzzy a-posteriori density*



EXAMPLE

$X \sim Ex_\theta\,, \quad \theta \in \Theta = (0,\infty)$

$$f(x|\theta) = \frac{1}{\theta}\exp\left\{-\frac{x}{\theta}\right\}\cdot I_{(0,\infty)}(x)$$

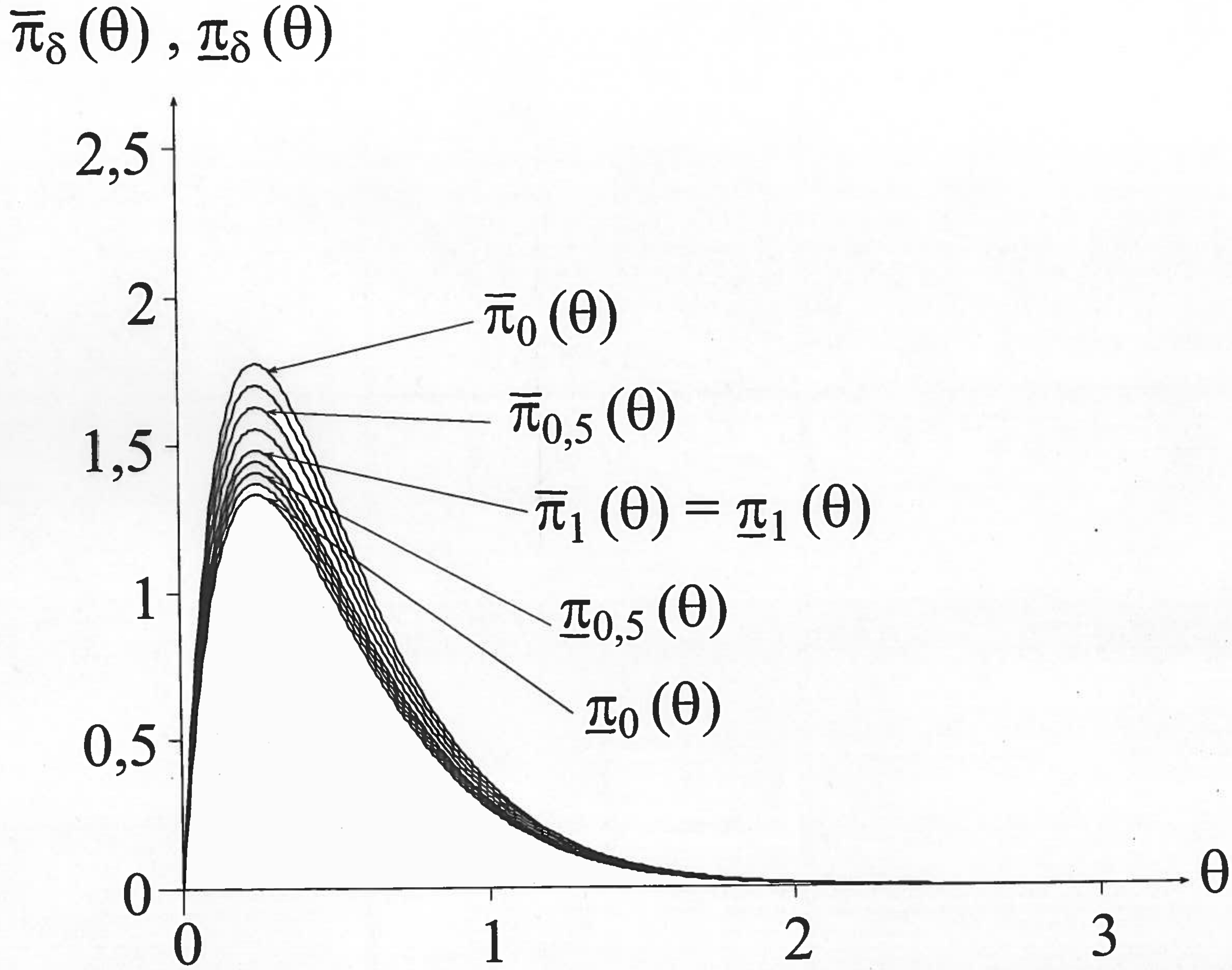
Fuzzy a-priori distribution

$\pi^*(\cdot)$ fuzzy gamma density

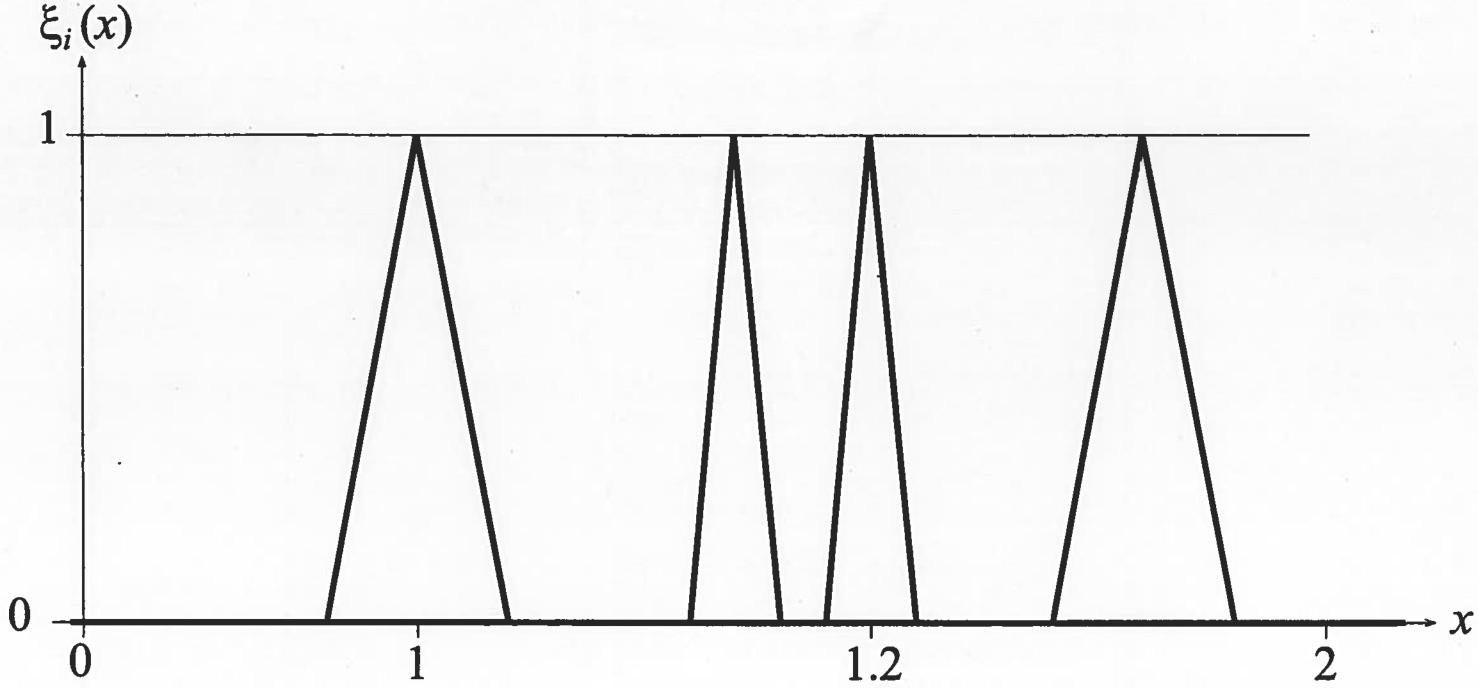
$\overline{\pi}_\delta(\cdot)$ upper

$\underline{\pi}_\delta(\cdot)$ lower

$\Big\}$ δ-level curves



$\overline{\pi}_\delta(\theta)$, $\underline{\pi}_\delta(\theta)$
2,5
2
1,5
1
0,5
0
$\overline{\pi}_0(\theta)$
$\overline{\pi}_{0,5}(\theta)$
$\overline{\pi}_1(\theta) = \underline{\pi}_1(\theta)$
$\underline{\pi}_{0,5}(\theta)$
$\underline{\pi}_0(\theta)$
θ
0
1
2
3



ξᵢ(x)
1
0
0
1
1.2
2
x

COMBINED FUZZY SAMPLE

$$\underline{x}^* = (x_1, x_2, x_3, x_4)^*$$

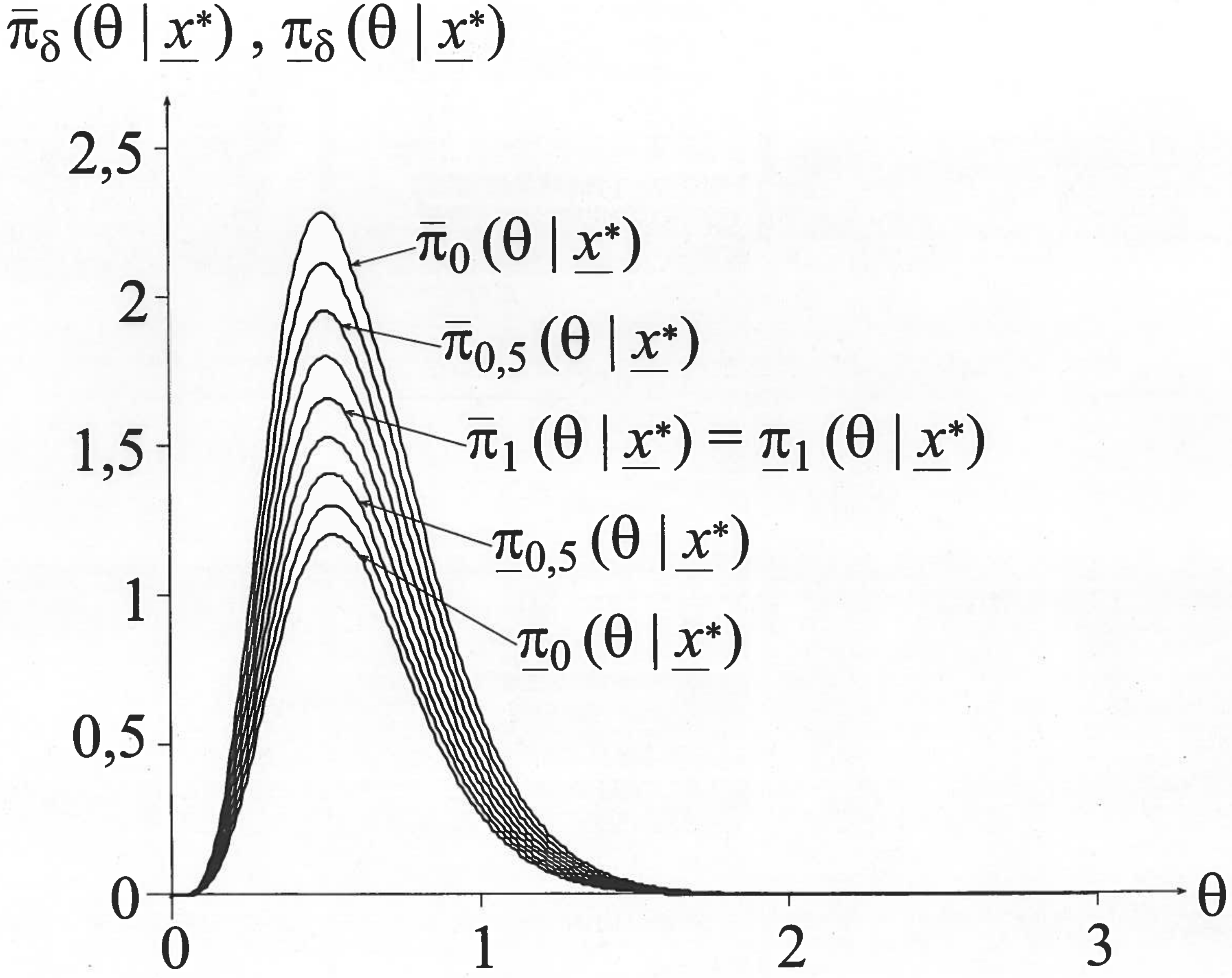
vector char. function $\xi(\cdot,\cdot,\cdot,\cdot)$

$$\xi(x_1, x_2, x_3, x_4) = \min\{\xi_1(x_1), \xi_2(x_2), \xi_3(x_3), \xi_4(x_4)\}$$

$\overline{\pi}_\delta(\cdot \mid \underline{x}^*)$

by gen. Bayes' theorem

$\underline{\pi}_\delta(\cdot \mid \underline{x}^*)$



$\bar{\pi}_\delta(\theta \mid \underline{x}^*)$, $\underline{\pi}_\delta(\theta \mid \underline{x}^*)$
2,5
2
1,5
1
0,5
0
$\bar{\pi}_0(\theta \mid \underline{x}^*)$
$\bar{\pi}_{0,5}(\theta \mid \underline{x}^*)$
$\bar{\pi}_1(\theta \mid \underline{x}^*) = \underline{\pi}_1(\theta \mid \underline{x}^*)$
$\underline{\pi}_{0,5}(\theta \mid \underline{x}^*)$
$\underline{\pi}_0(\theta \mid \underline{x}^*)$
0
1
2
3
θ

PREDICTIVE DENSITIES

$X \sim f(\cdot|\theta)$, $\theta \in \Theta$ Stochastic Model

$\pi(\cdot)$ a-priori density

$(x_1, \cdots, x_n) = D$ data

$\Rightarrow$ $\pi(\cdot|D)$ a-posteriori density

$p(\cdot|D)$ predictive density

$$p(x|D) = \int_{\Theta} f(x|\theta) \cdot \pi(\theta|D)\, d\theta \quad \forall x \in M_X$$

FUZZY PREDICTIVE DENSITY

$$p^*(\cdot \mid D^*)$$

$$p^*(x \mid D^*) = \int_{\Theta} f(x \mid \theta) \odot \pi^*(\theta \mid D^*)\, d\theta \qquad \forall x \in M_X$$

$$\mathcal{D}_\delta := \{ g(\cdot) \text{ density on } \Theta : \underline{\pi}_\delta(\theta) \le g(\theta) \le \overline{\pi}_\delta(\theta) \ \ \forall \theta \in \Theta \}$$

$$a_\delta := \inf \left\{ \int_{\Theta} f(x \mid \theta)\, g(\theta)\, d\theta : g(\cdot) \in \mathcal{D}_\delta \right\}$$

$$\forall \delta \in (0;1]$$

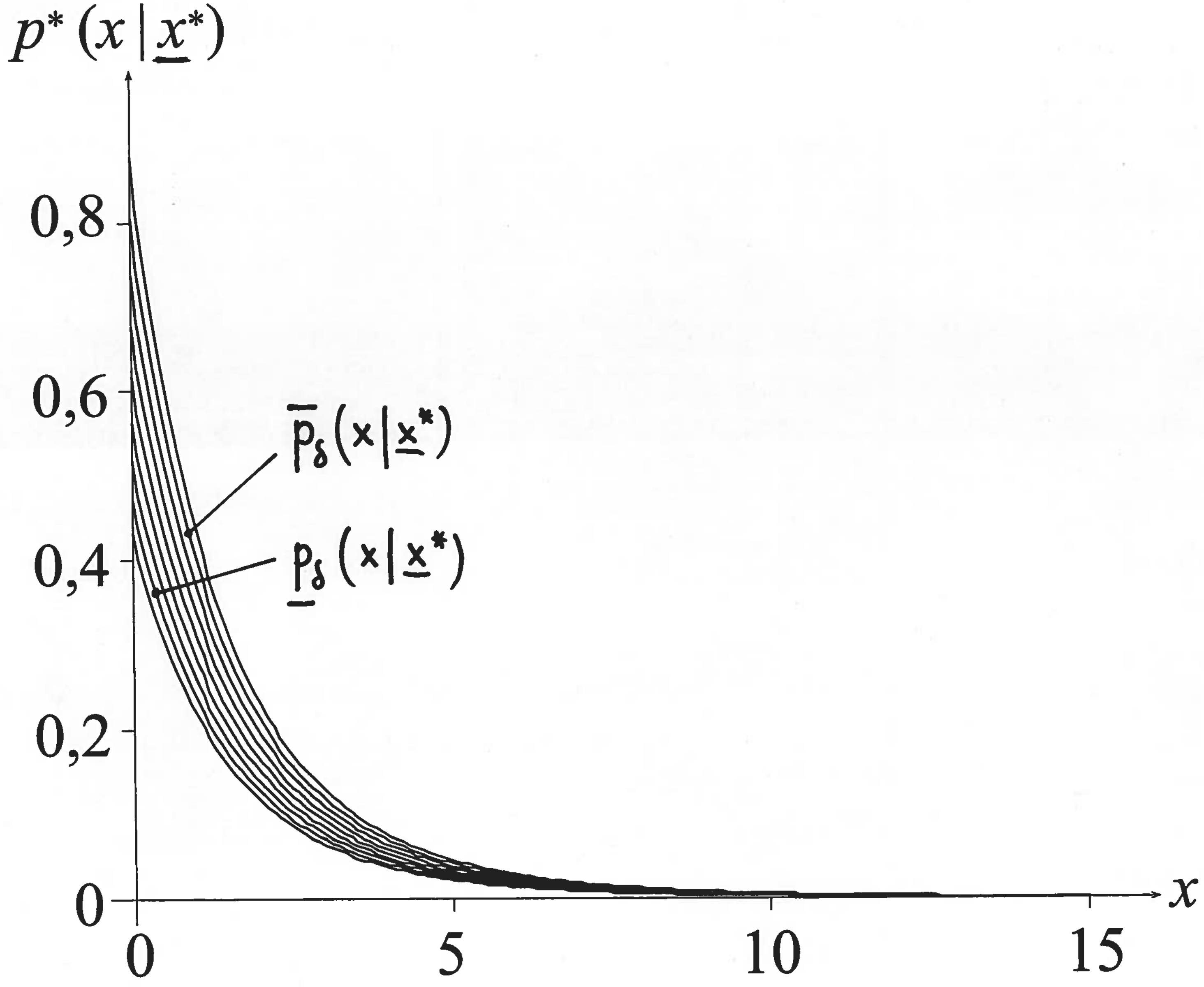
$$b_\delta := \sup \left\{ \int_{\Theta} f(x \mid \theta)\, g(\theta)\, d\theta : g(\cdot) \in \mathcal{D}_\delta \right\}$$

The nested family of intervals $[a_\delta; b_\delta]$ defines a fuzzy number by the construction lemma:

$$\psi_x(y) := \sup\left\{ \delta \cdot \mathbb{1}_{[a_\delta; b_\delta]}(y) : \delta \in [0;1] \right\} \quad \forall y \in \mathbb{R}$$

$$p^*(x \mid D^*) \mathrel{\hat{=}} \psi_x(\cdot)$$

For variable x this is a <u>fuzzy density</u>



p* (x | x*)
0,8
0,6
0,4
0,2
0
p̄_δ (x | x*)
p_δ (x | x*)
0
5
10
15
x

CONCLUSIONS

- Fuzziness can be described quantitatively
- Statistics based on fuzzy information is possible: Two different uncertainties
- Kolmogorov's probability concept has to be generalized
- Hybrid approach: Fuzzy and Stochastics

SOFTWARE

- Some Programs

  C++, R

- Under Development:

  SAFD, ECSC at Mieres

SOME REFERENCES

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