Streamlining the Applied Mathematics Studies at Faculty of Science of Palacký University in Olomouc
CZ.1.07/2.2.00/15.0243

evropský sociální fond v ČR

EVROPSKÁ UNIE

MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY

OP Vzdělávání pro konkurenceschopnost

INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ

International Conference
Olomoucian Days of Applied Mathematics

ODAM 2013

Department of Mathematical analysis
and Applications of Mathematics
Faculty of Science
Palacký Univerzity Olomouc

Negotiation, Bargaining, Arbitration

Milan Vlach

Faculty of Mathematics and Physics, Charles University
Czech Republic

Institute of Information and Automation, Academy of Sciences
Czech Republic

Kyoto College of Graduate Studies for Informatics
Japan

The formulation of a problem is often more essential than its solution, which may be merely a matter of mathematical or experimental skill.

Einstein, Infeld

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Einstein, Infeld

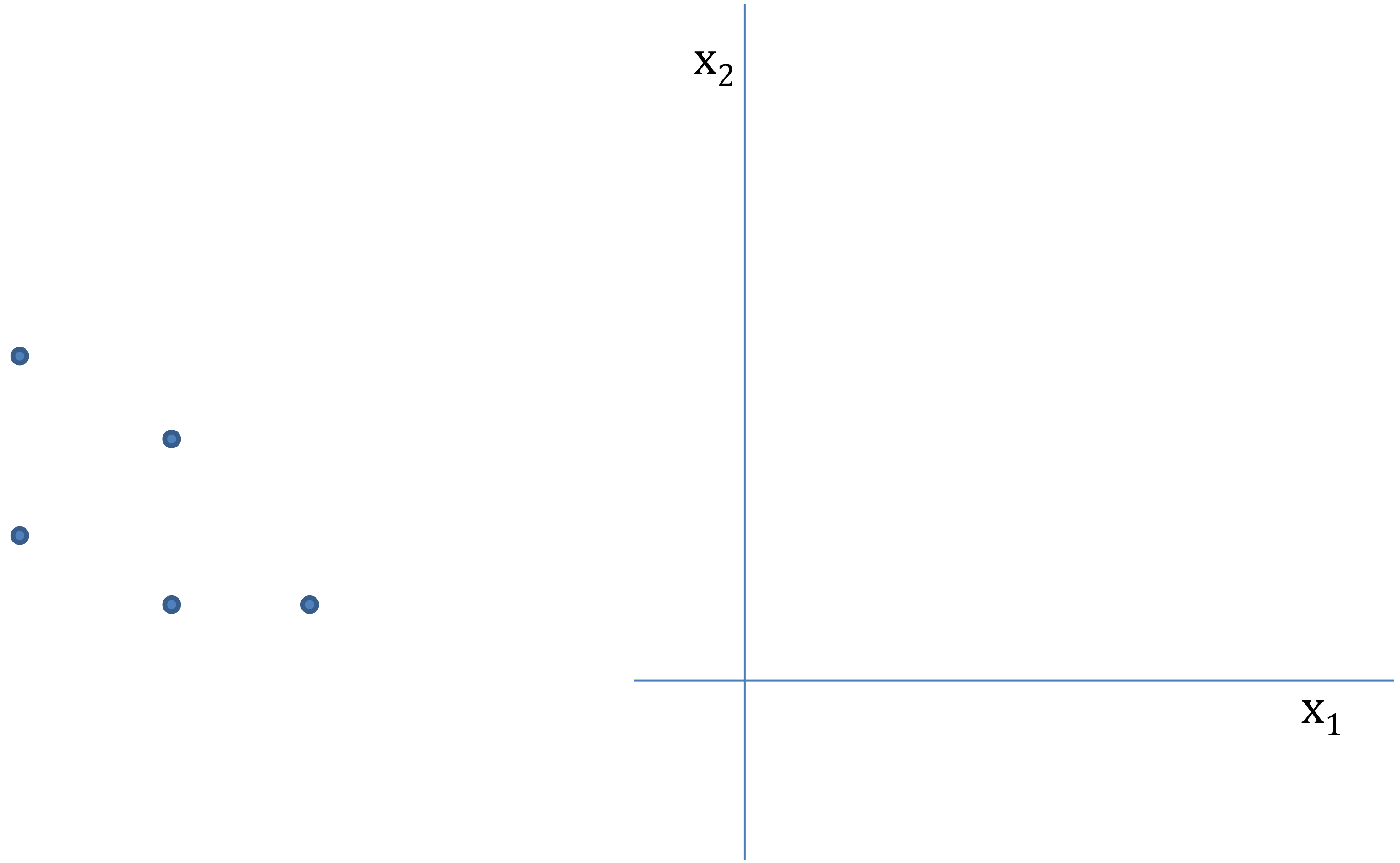
In a formal model the conclusions are derived from definitions and assumptions. But with informal verbal reasoning, one can argue until one is blue in the face, because there is no criterion for deciding the soundness of an informal argument.

Robert Aumann

The problem we are interested in concerns the situations where:

- A given number of individuals (bargainers, players, . . .) have to reach a unanimous decision about which alternative to select from a given set of feasible alternatives.

- If no unanimous agreement is reached, then certain pre-specified alternative will be selected.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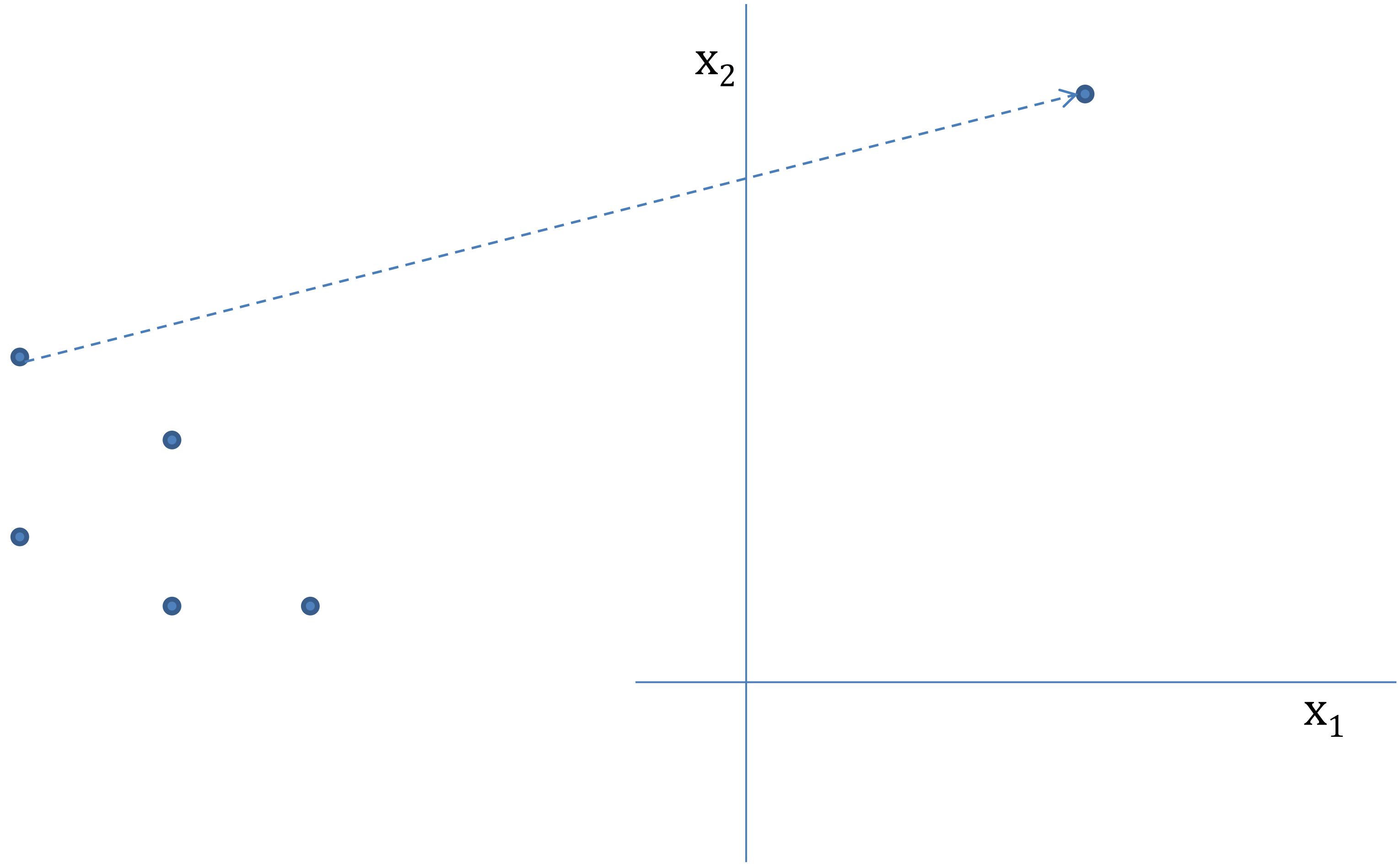

x_2
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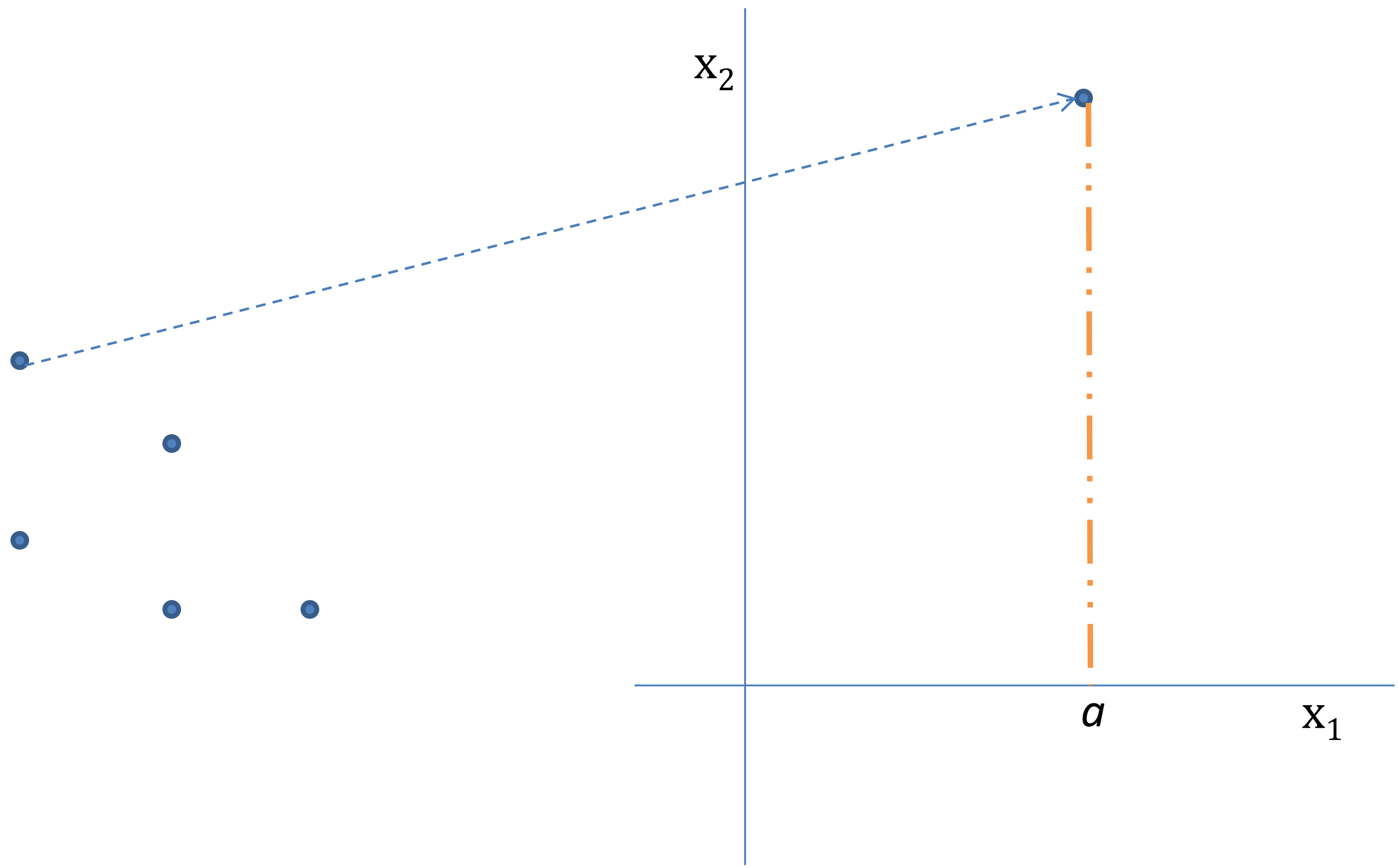

x_2
a
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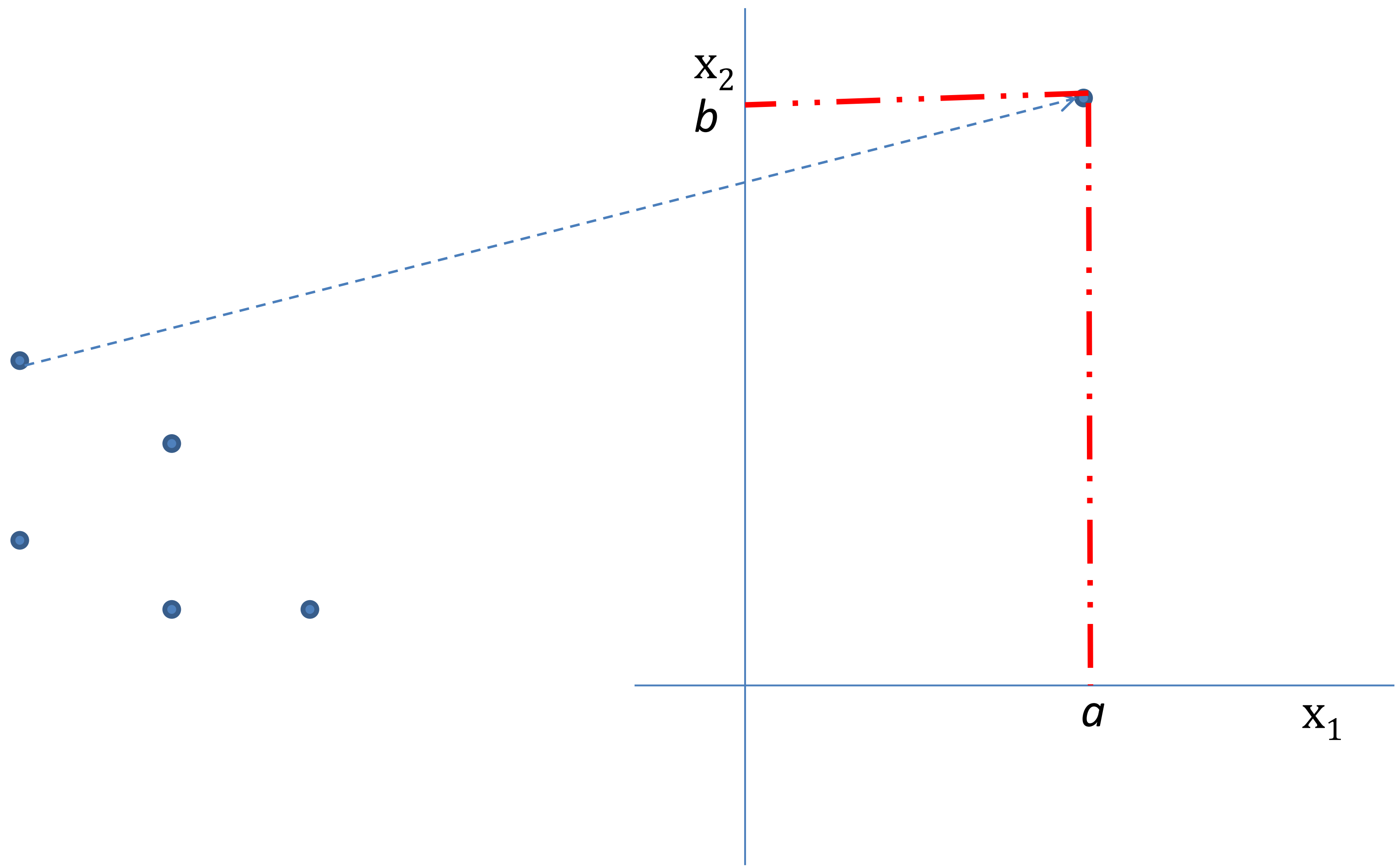
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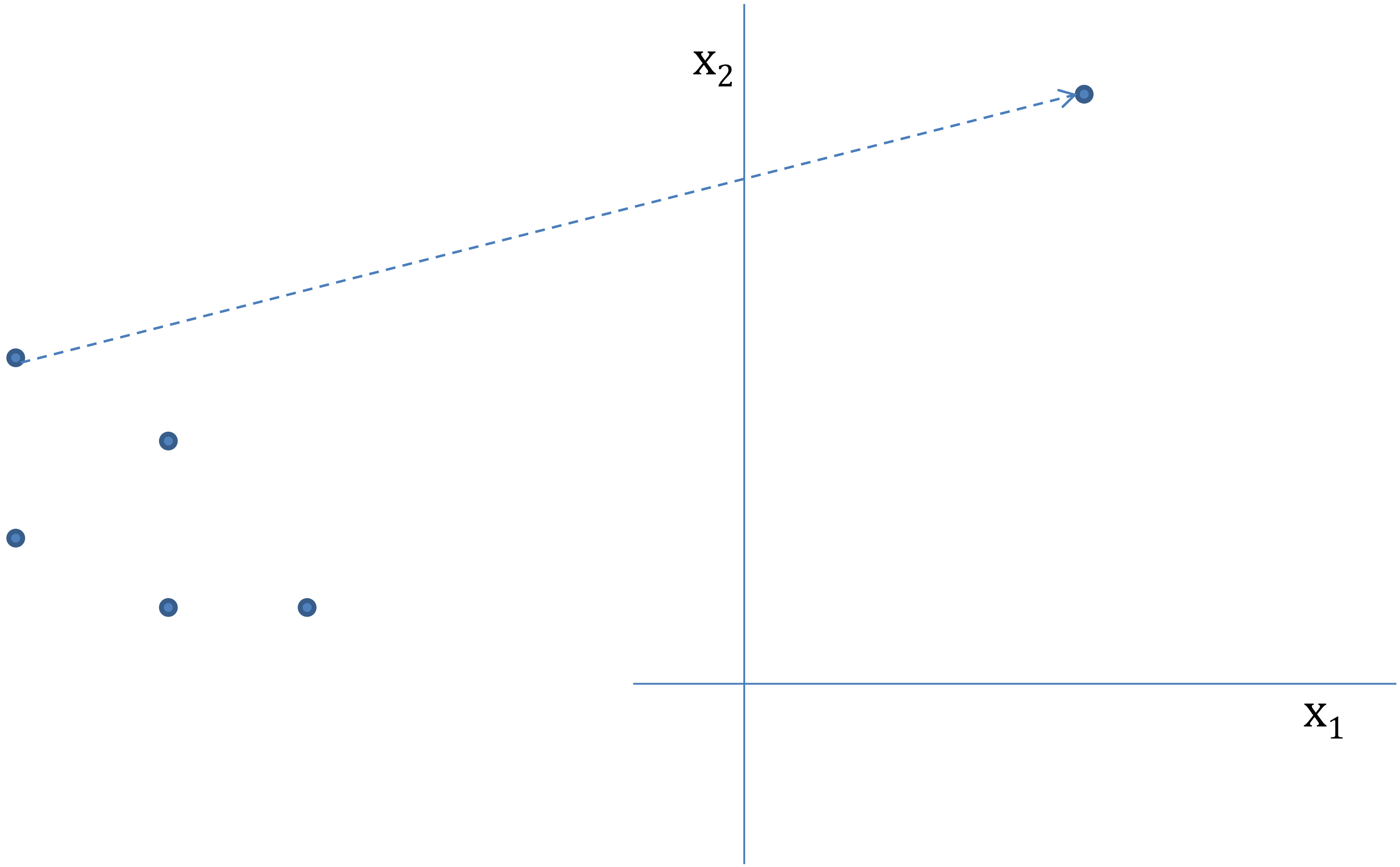
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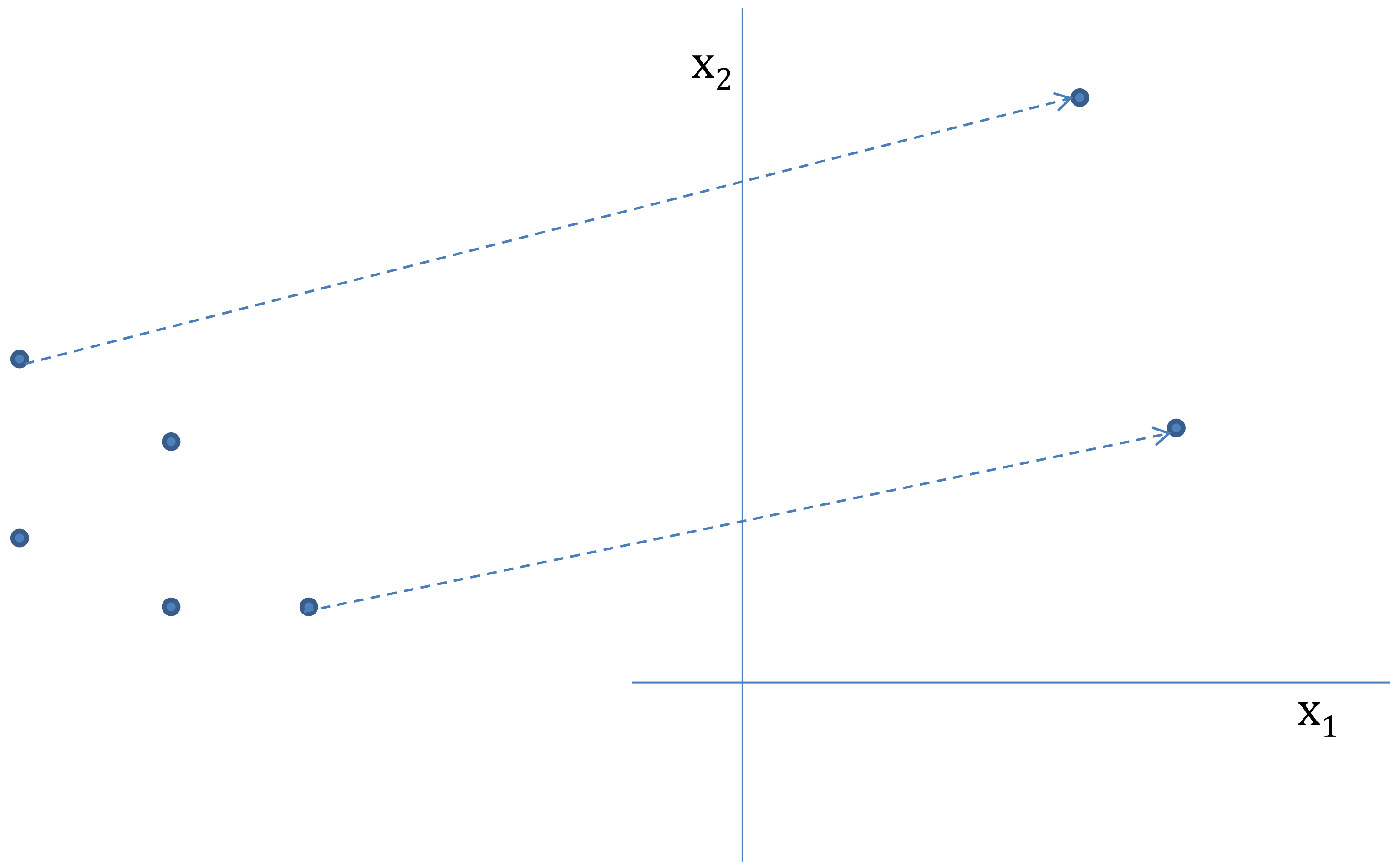
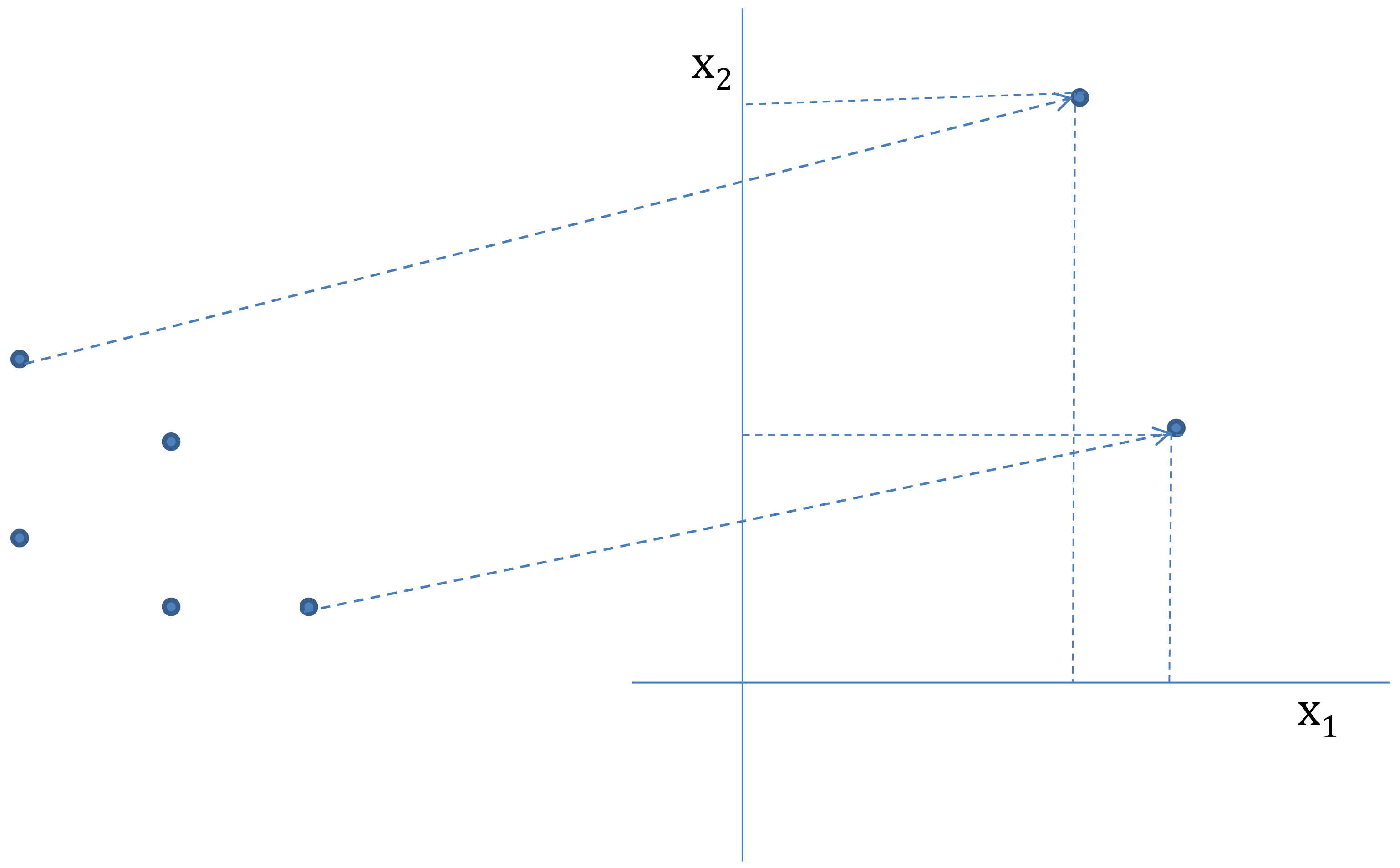


x_2
x_1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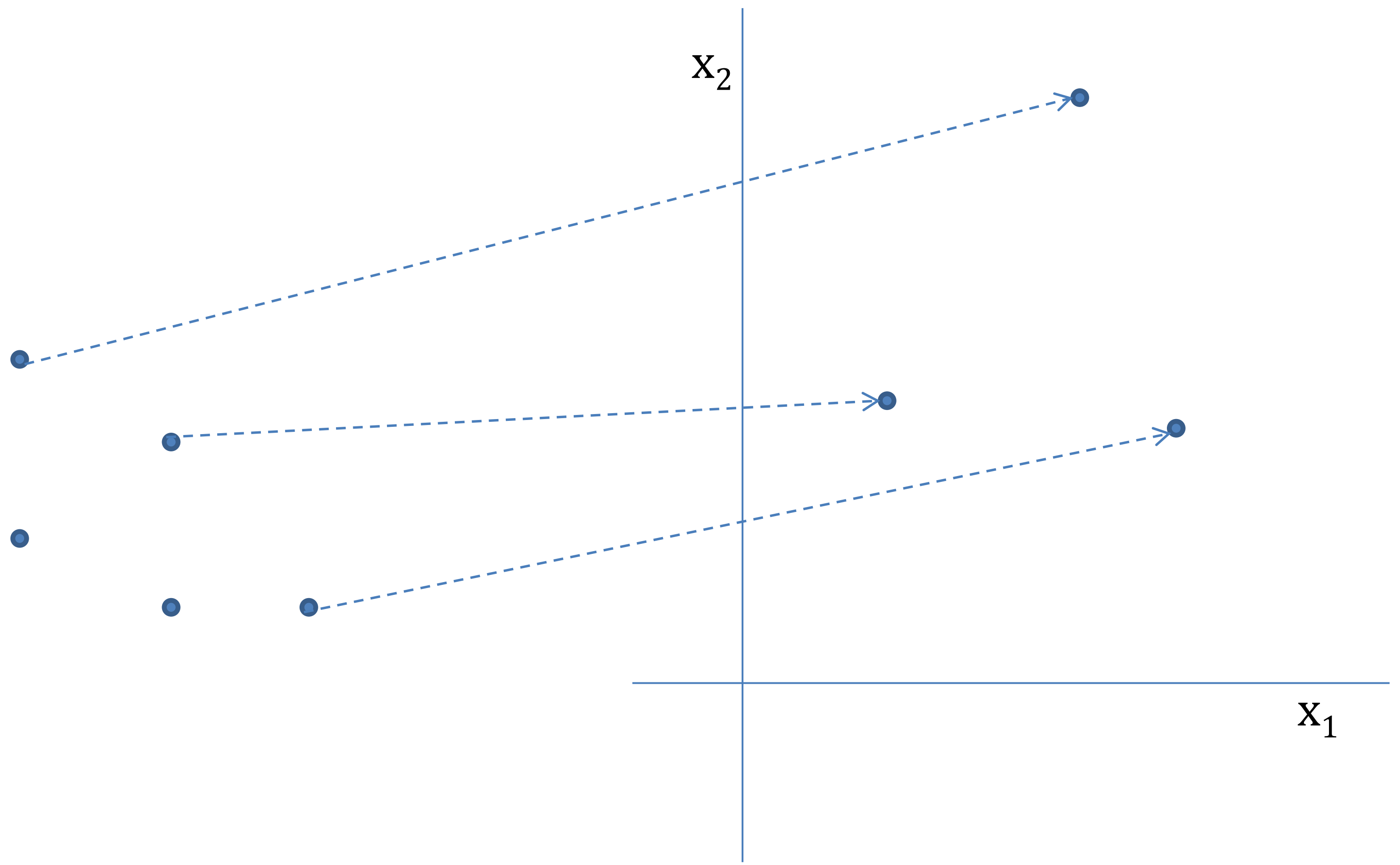
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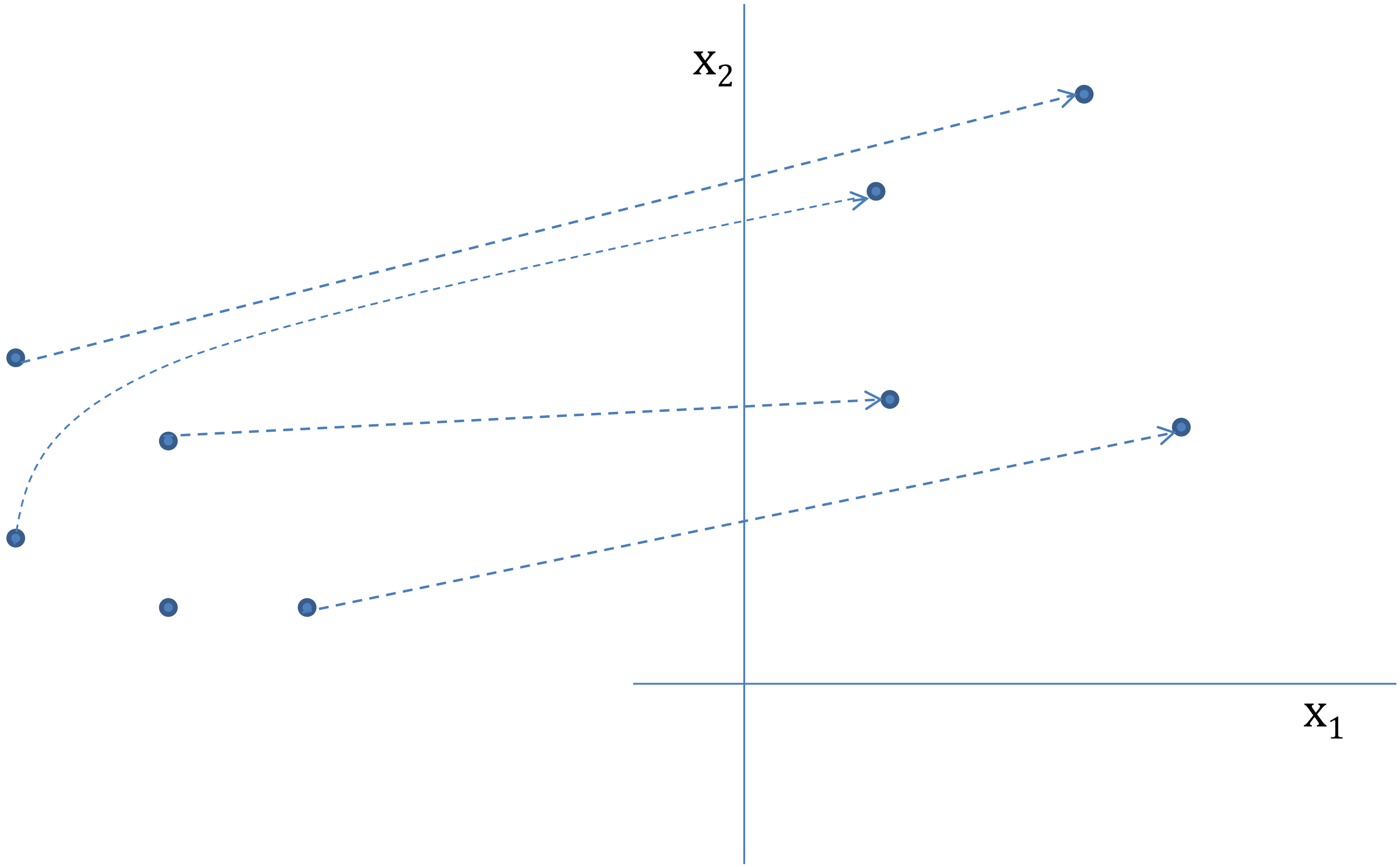
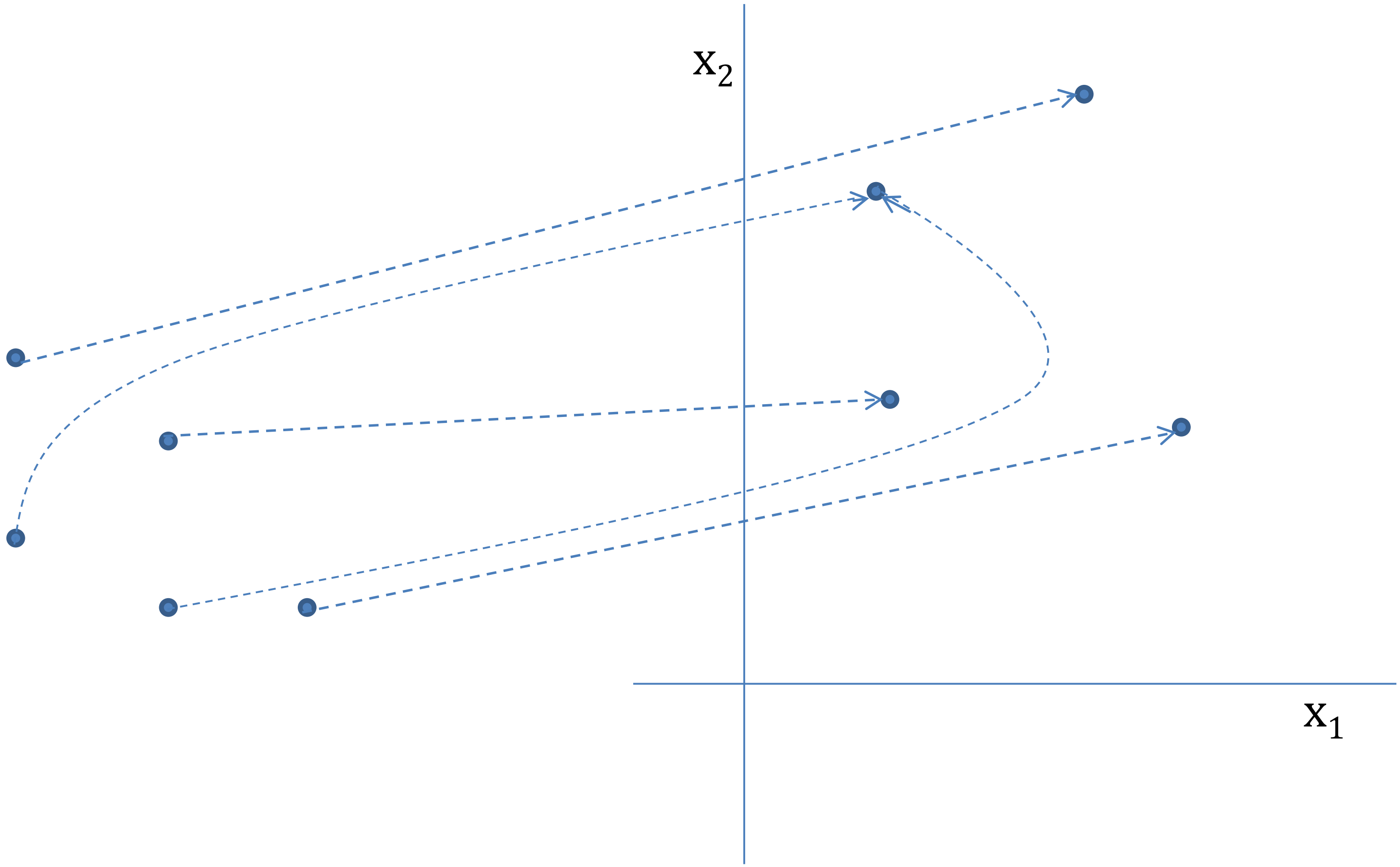
x_1



x_2
x_1



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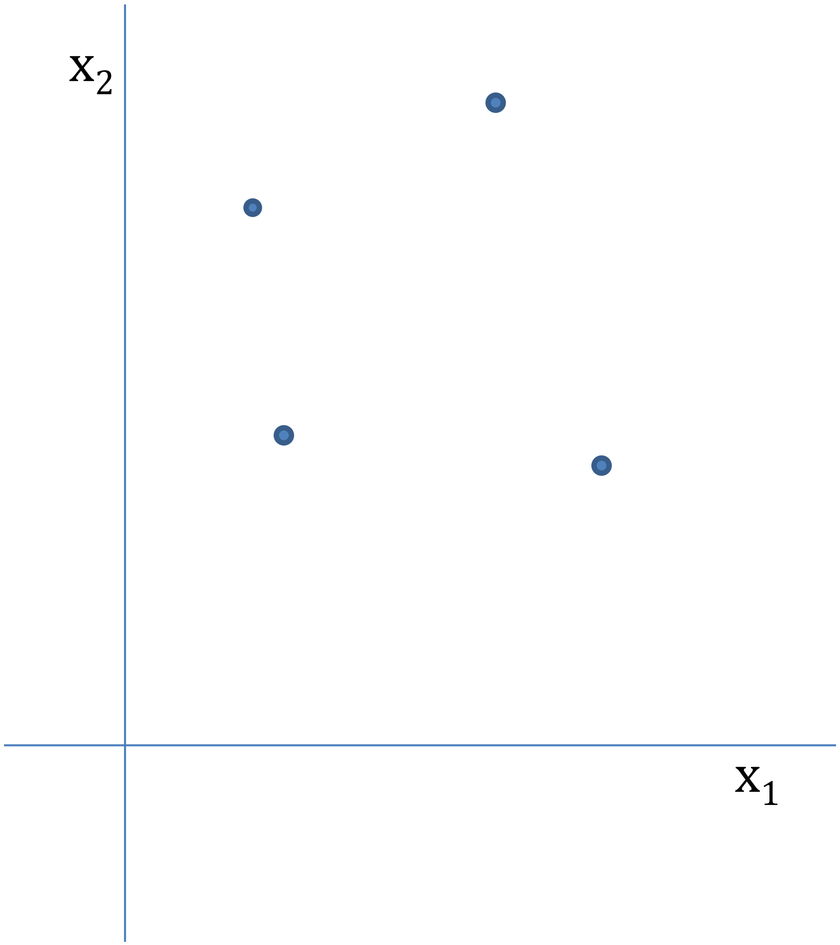
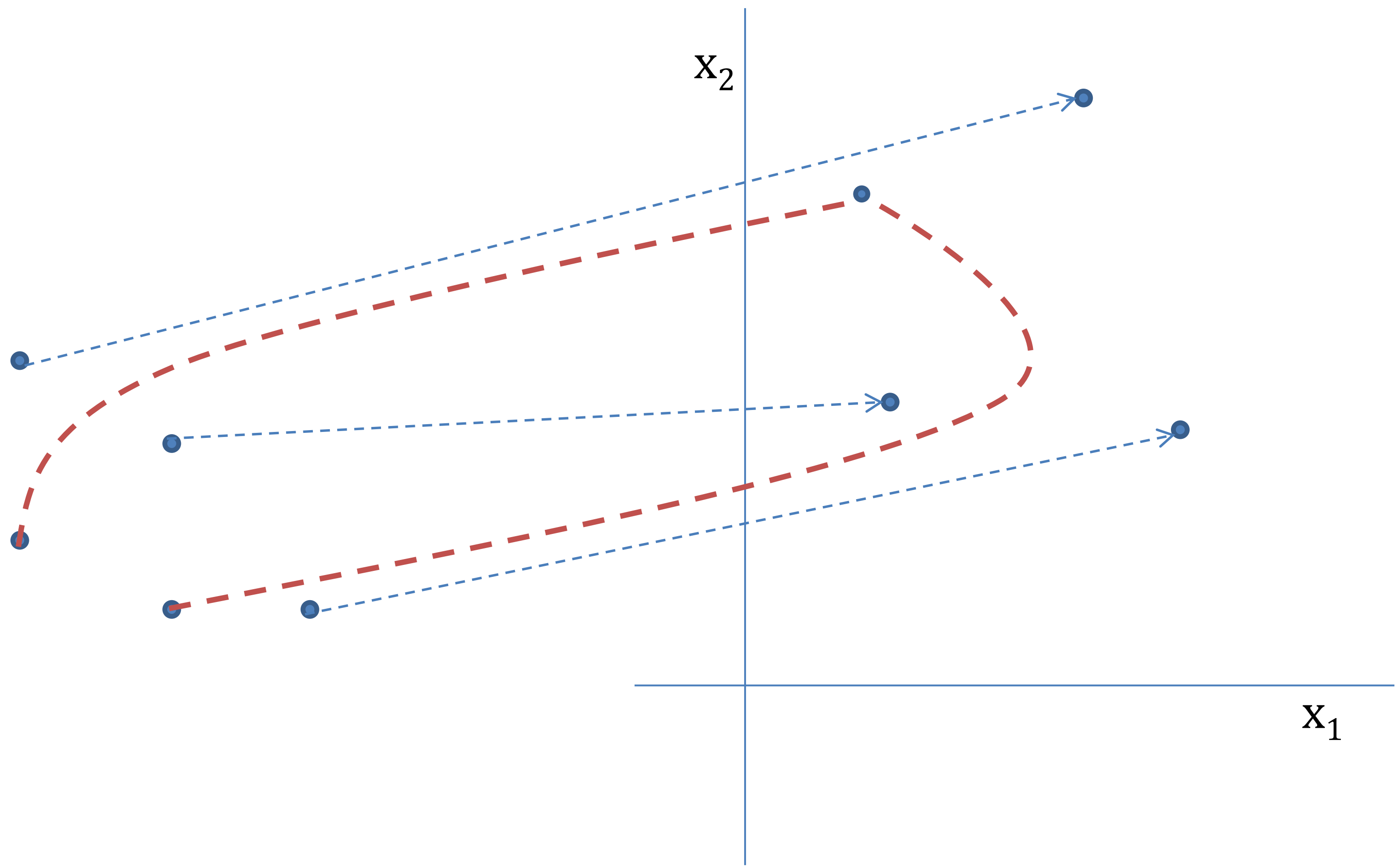
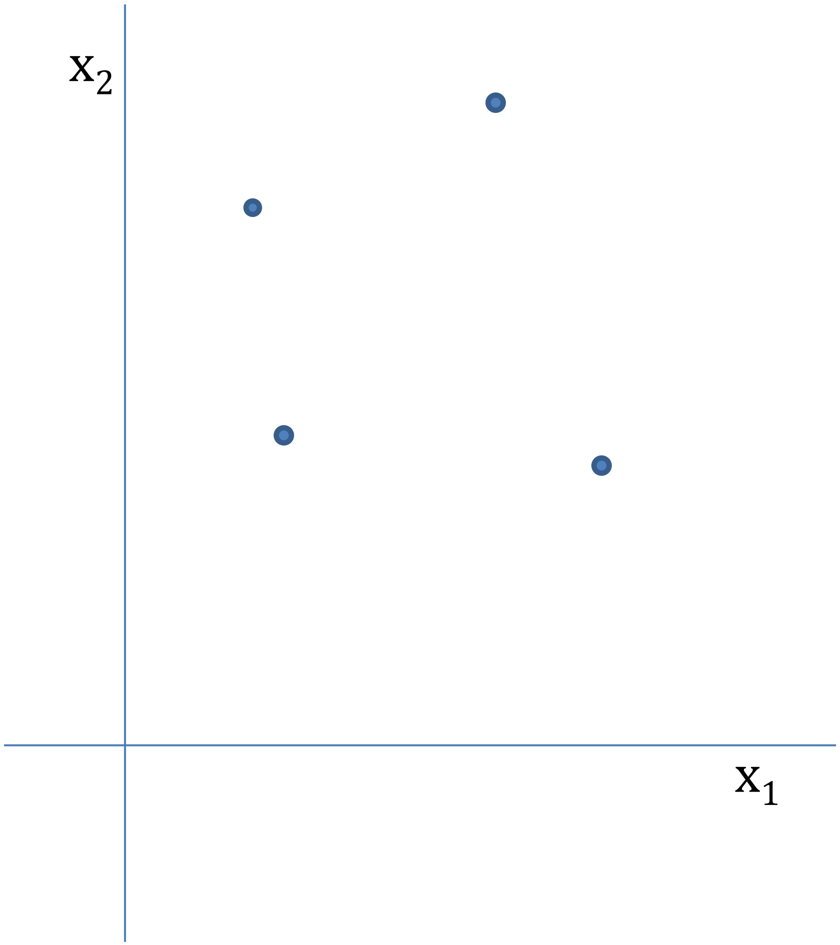
x_1



x_2
x_1

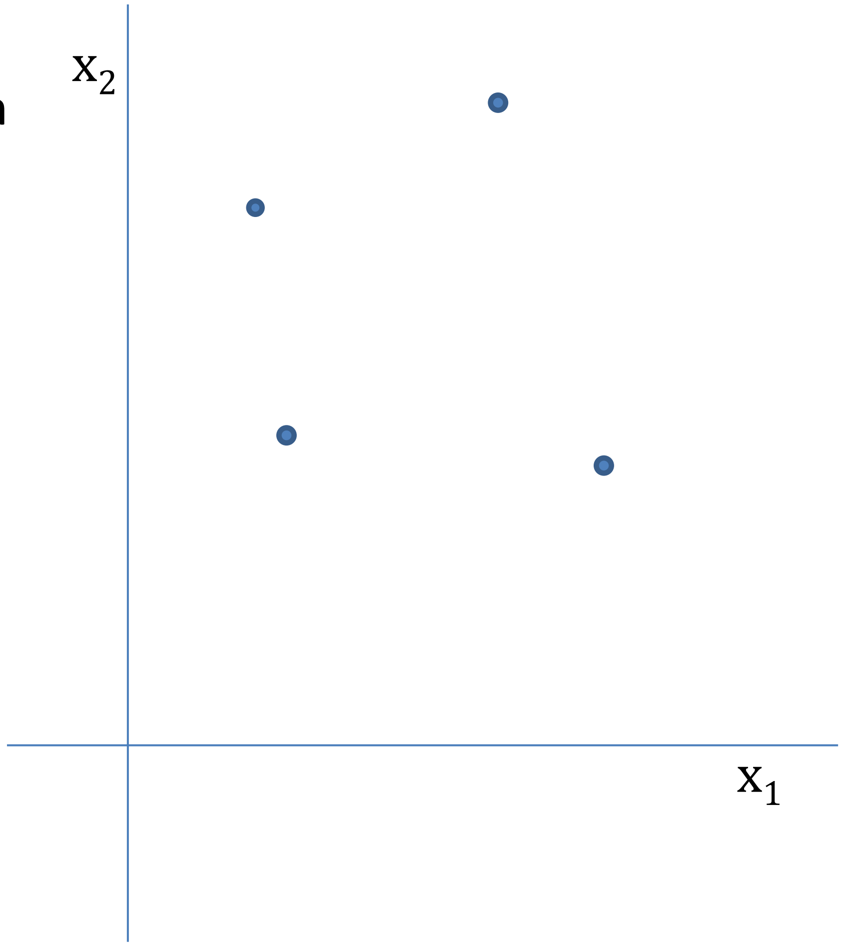


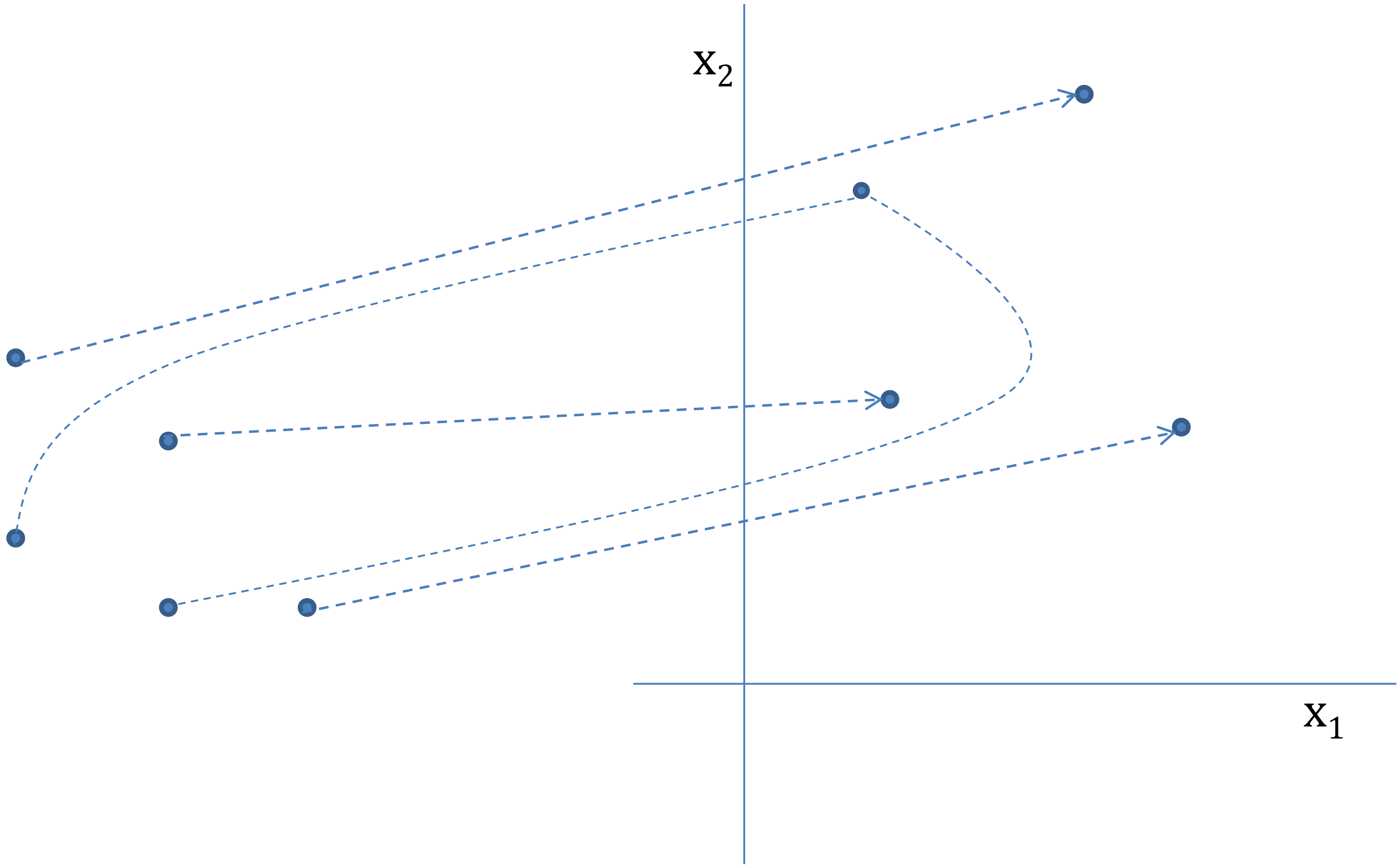
x_2
x_1



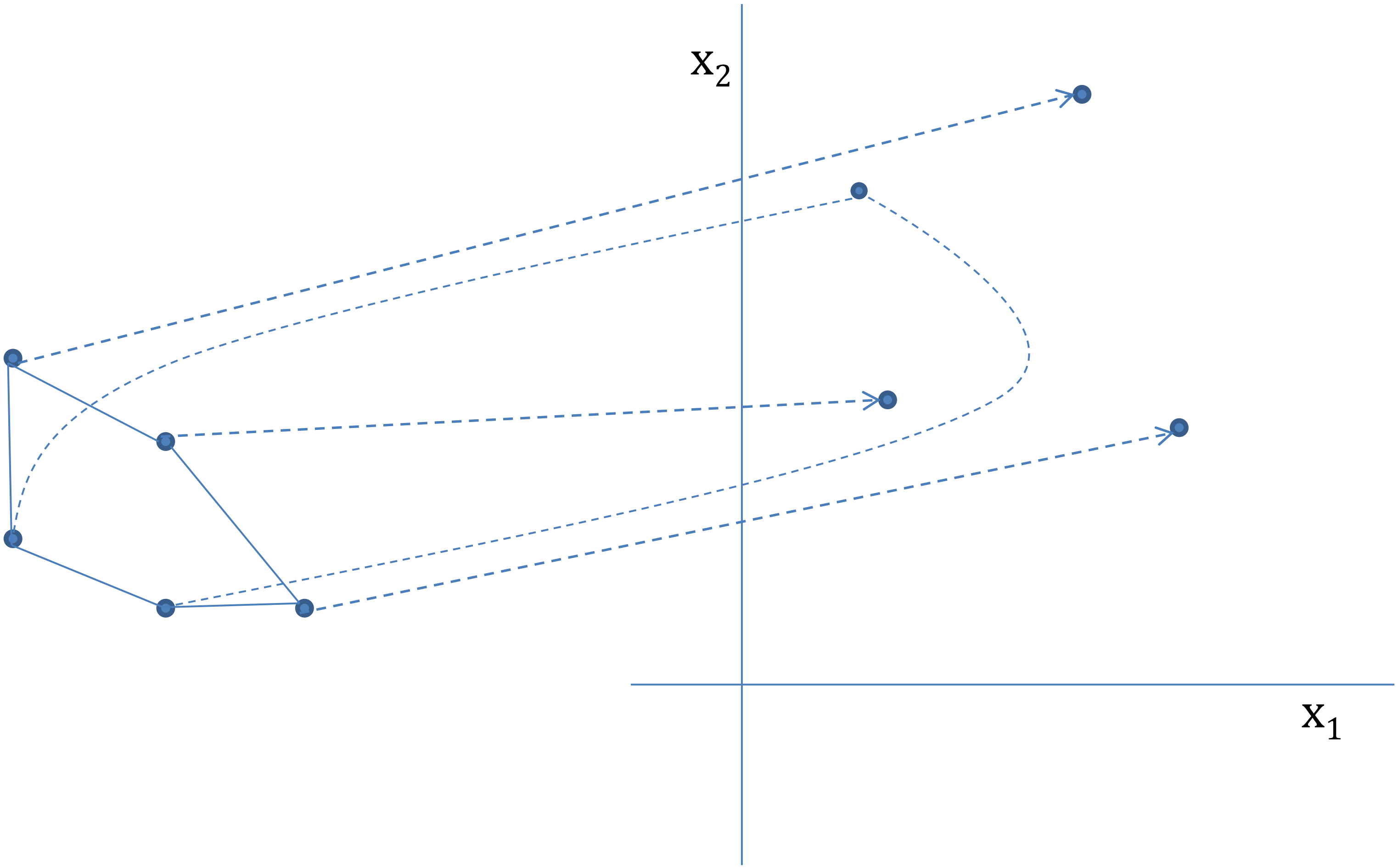
x_2
x_1

Often it is assumed that randomization on different alternatives is possible.

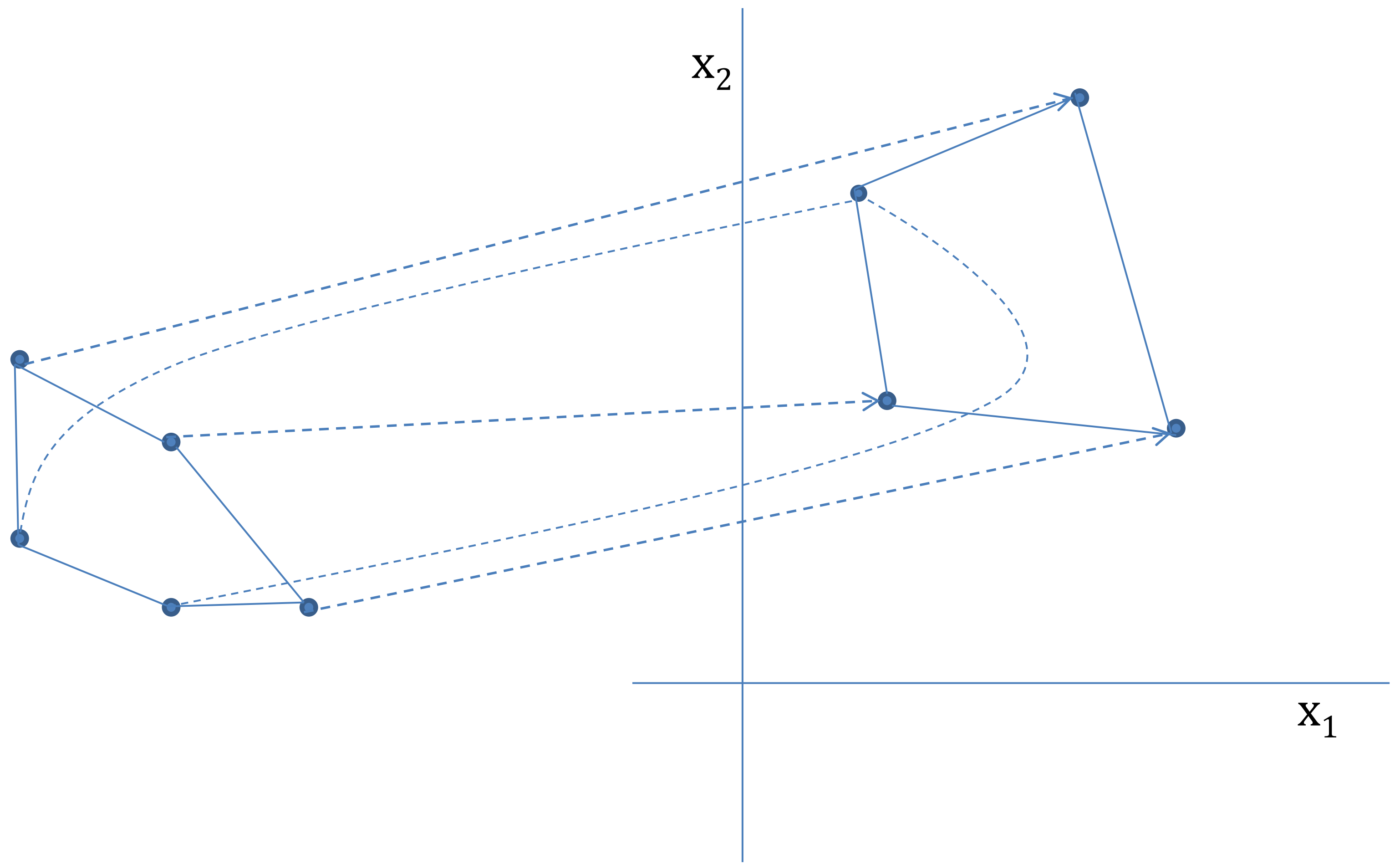




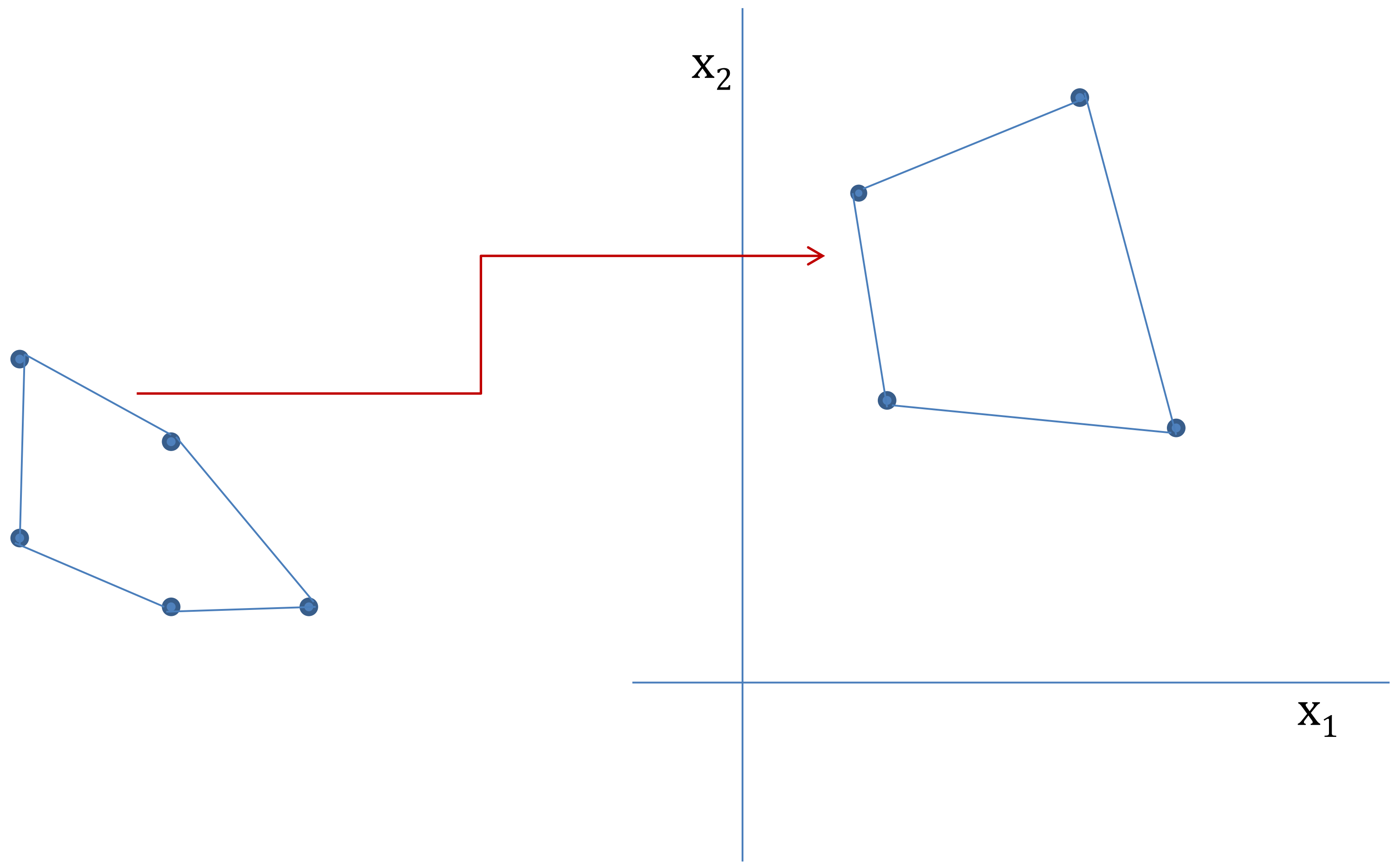
x_2
x_1



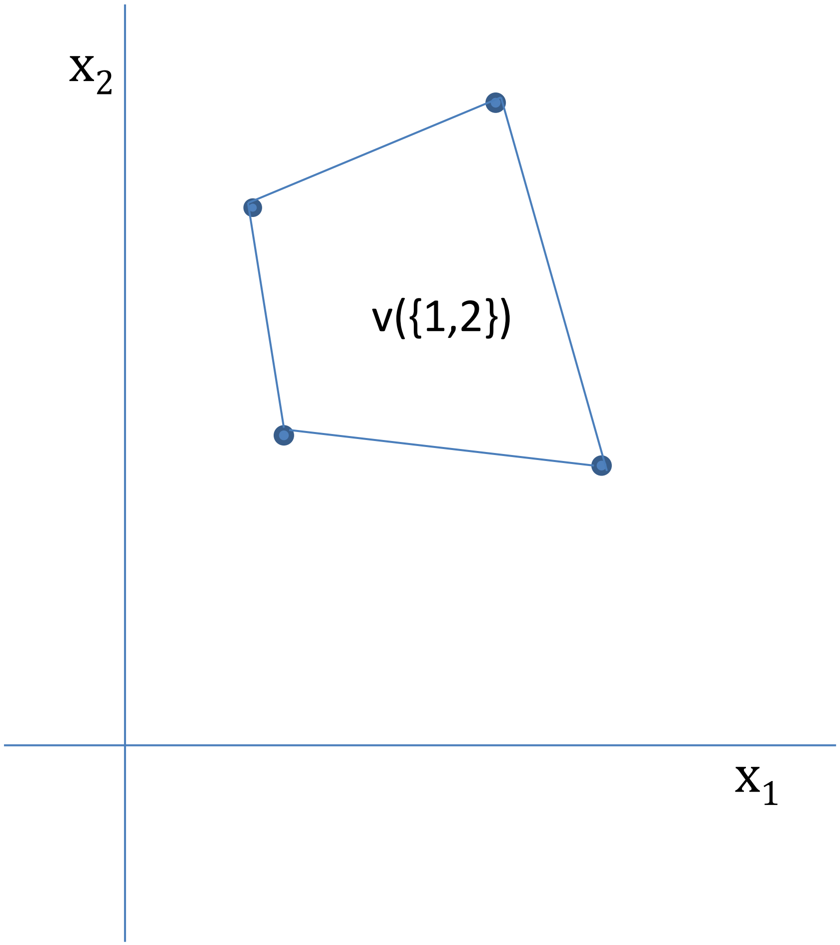
x_2
x_1

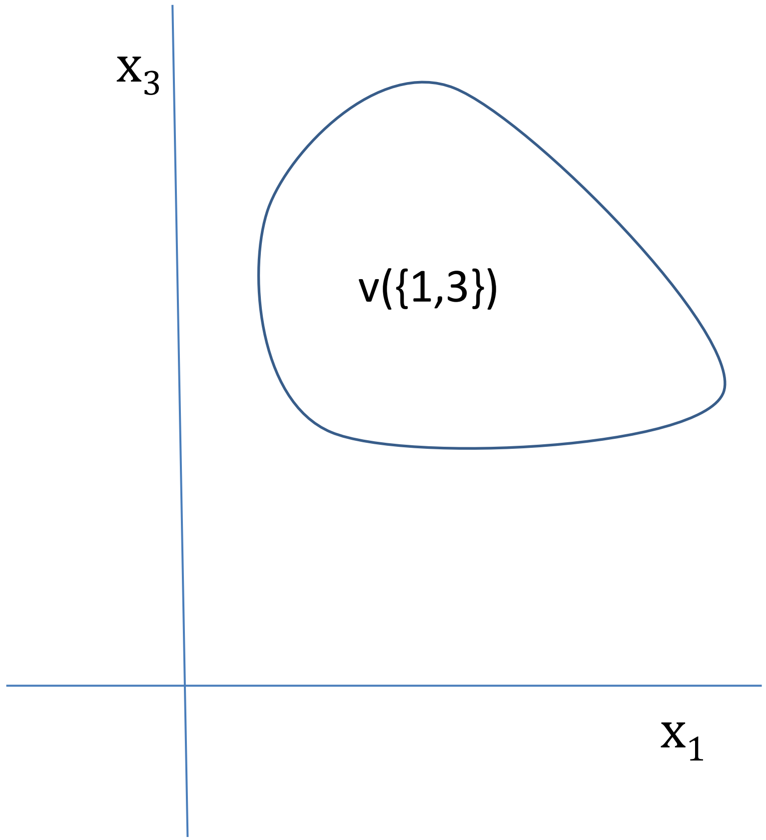


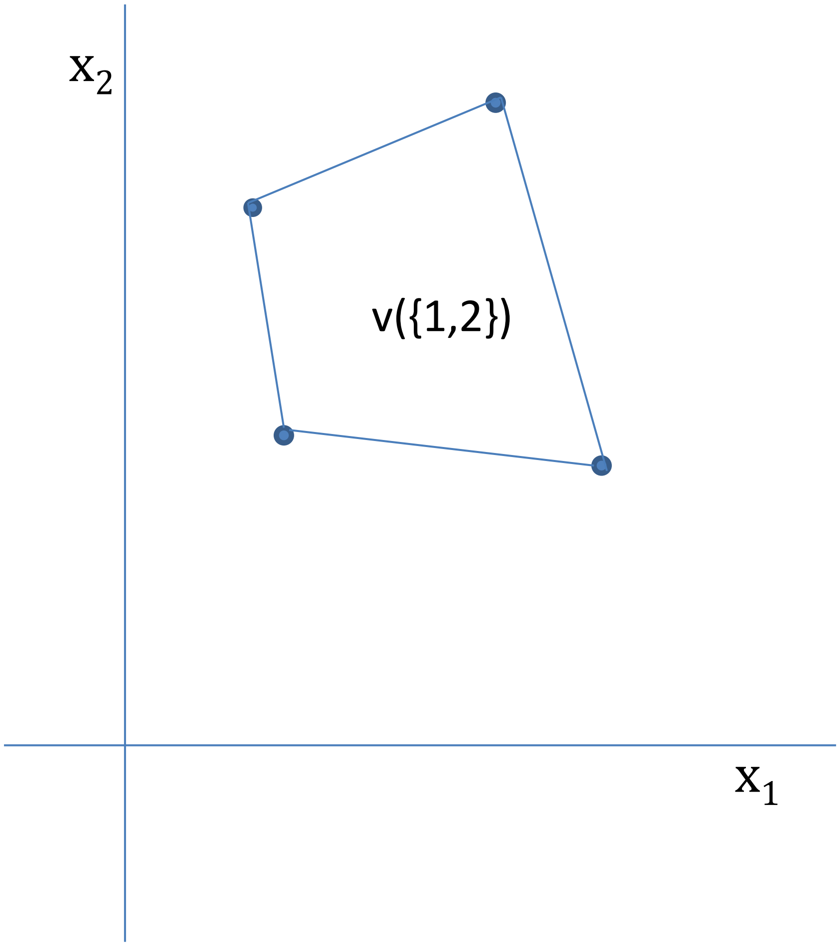
x_2
x_1

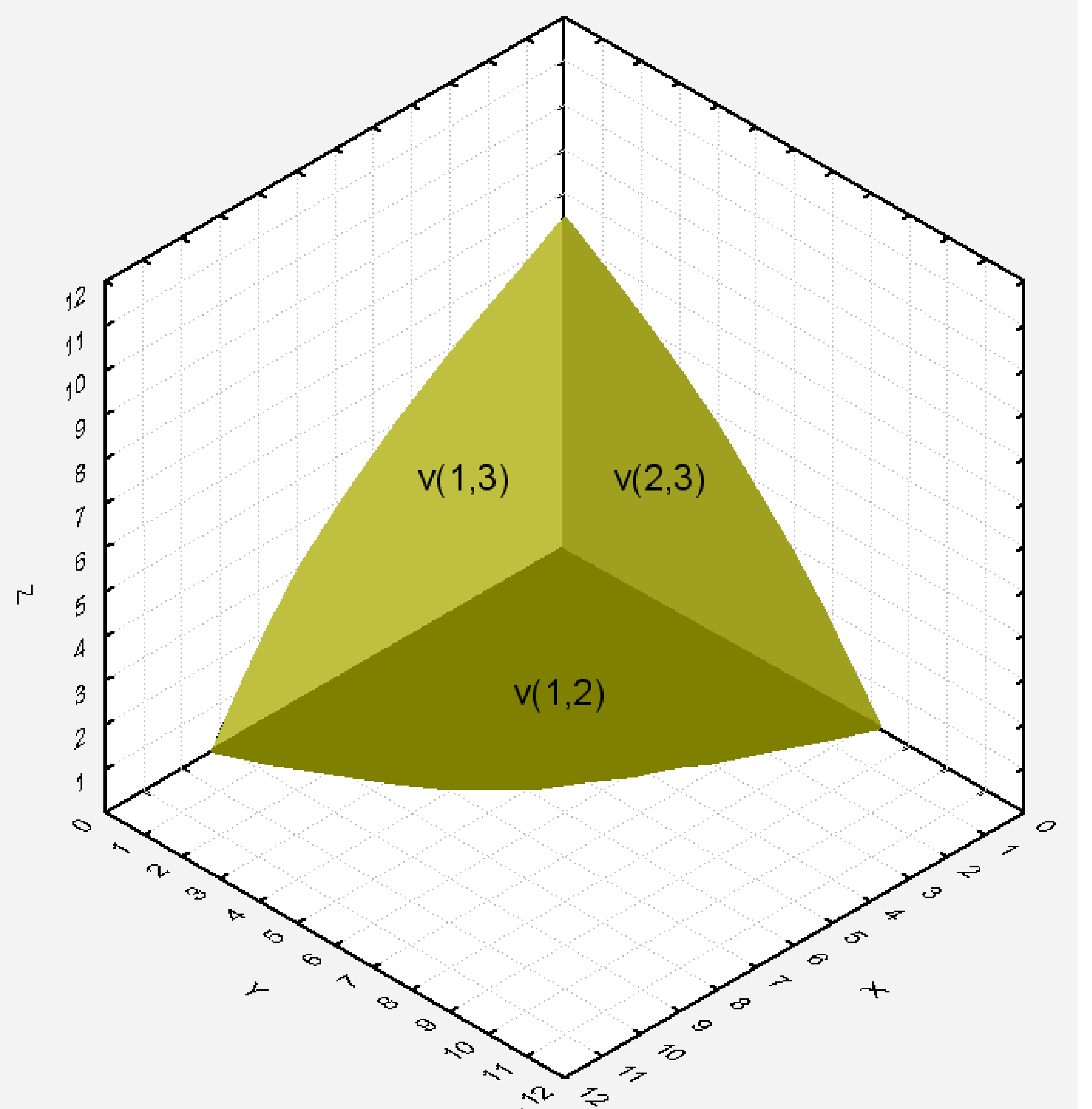


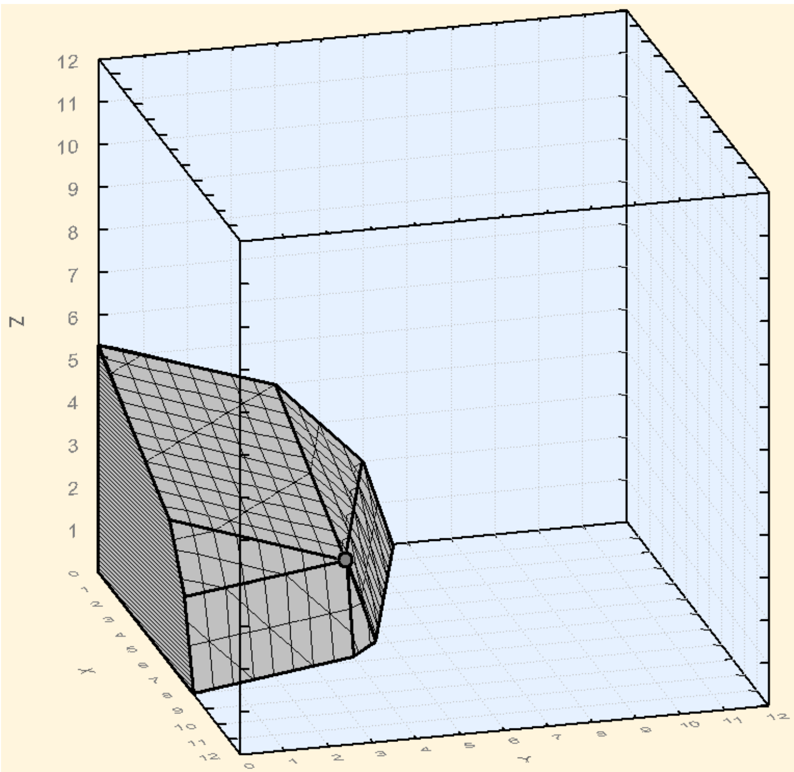
x_2
x_1


x2
v({1,2})
x1

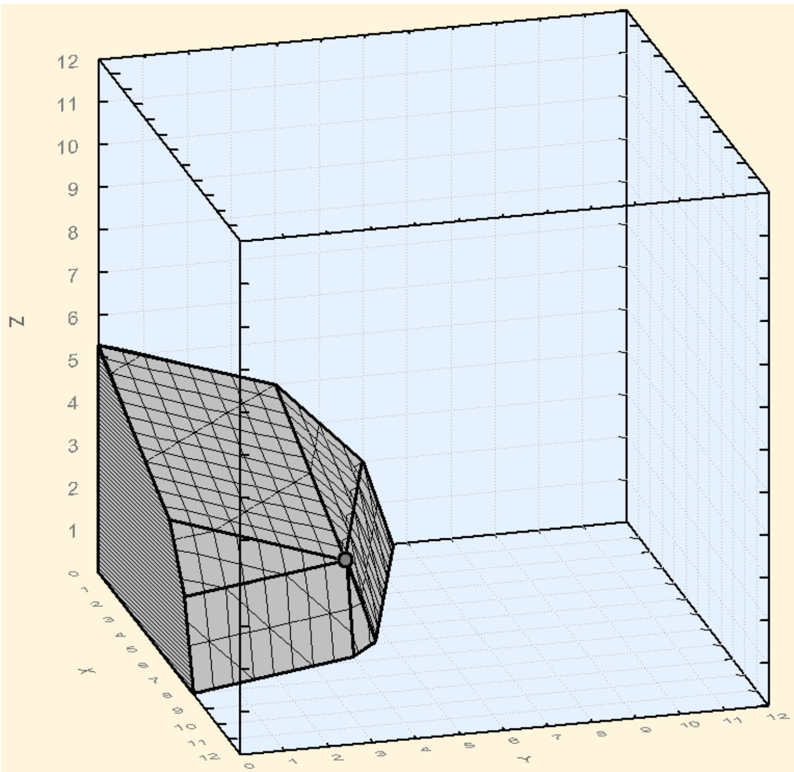

x3
v({1,3})
x1


x2
v({1,2})
x1


v(1,3)
v(2,3)
v(1,2)
Z
Y
X
0
1
2
3
4
5
6
7
8
9
10
11
12



Z
X
Y



Z
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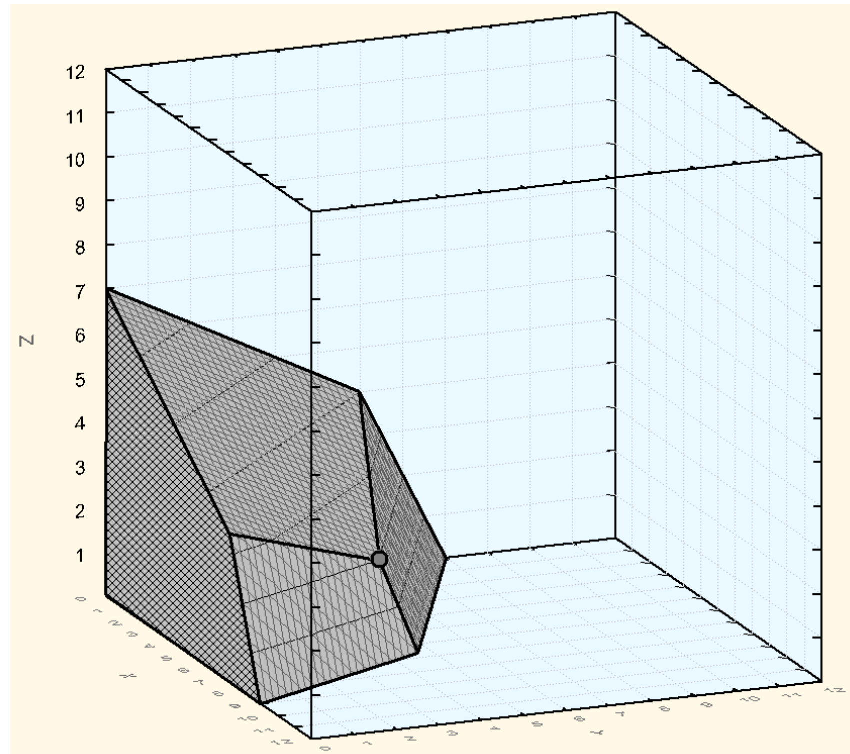

Z
X
Y

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- A given number of individuals (bargainers, players, . . .) have to reach a unanimous decision about which alternative to select from a given set of feasible alternatives.

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We shall formulate such bargaining problems in the framework of cooperative games.

Coalitional games

A *cooperative game in coalitional form* or briefly a *coalitional game* or just a *game* consists of

- nonempty sets N (set of players) and X (space of players' payoff profiles),
- a mapping v that assigns to every nonempty subset K of N (coalition) a subset $v(K)$ of X, and
- a family $\{\succ_i\}_{i\in N}$ of binary relations on X (players' preference relations).

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- The number of players is finite and greater than 1. We set $N = \{1, 2, \ldots, n\}, n \geq 2$.

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In these situations, we denote such an n-player coalitional game by (N, v) or simply v.

Bargaining games

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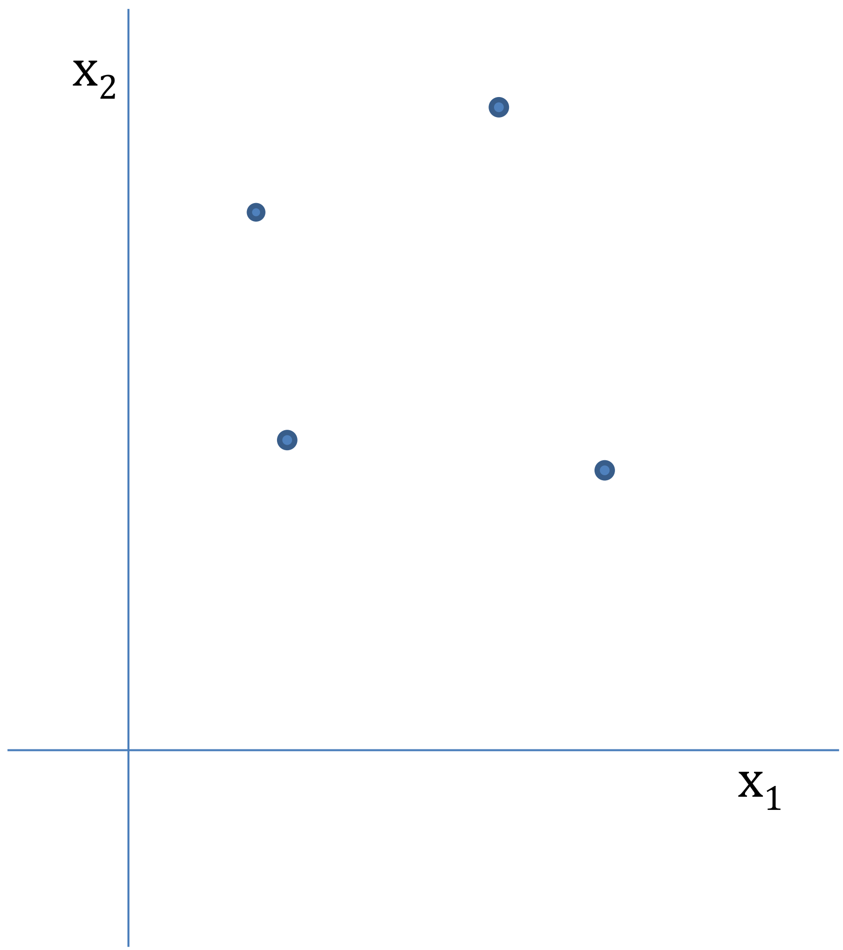
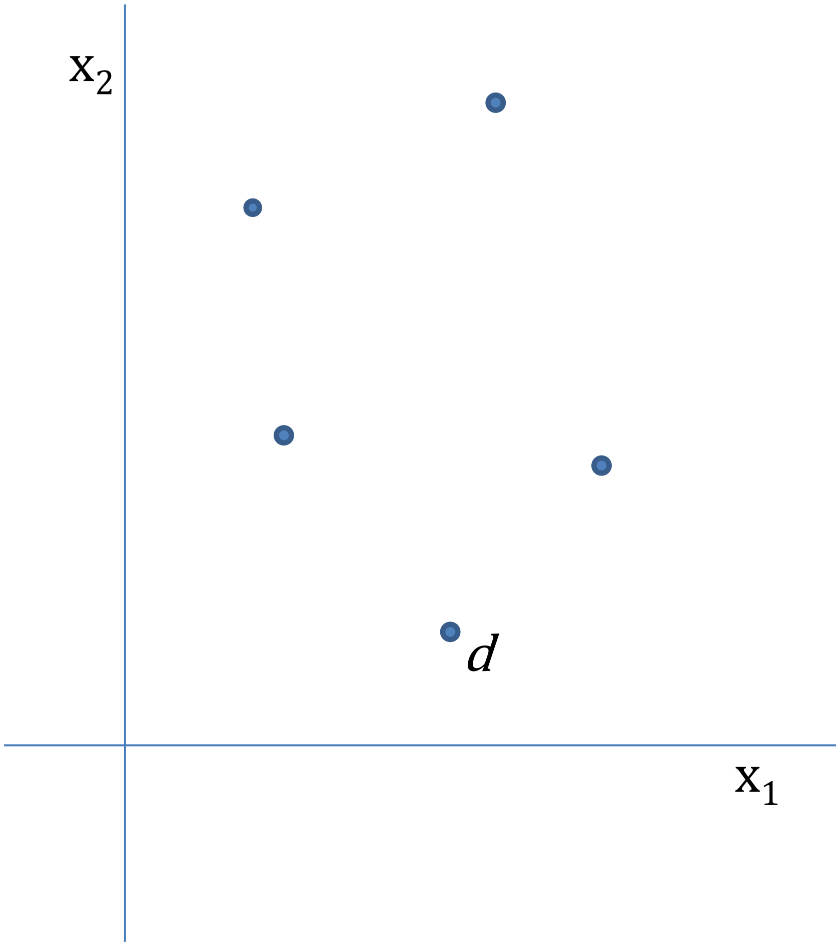
For an n-player bargaining game (N, v), it is customary to denote $v(N)$ by S, and the maximum payoff in $v(\{i\})$ by d_i.

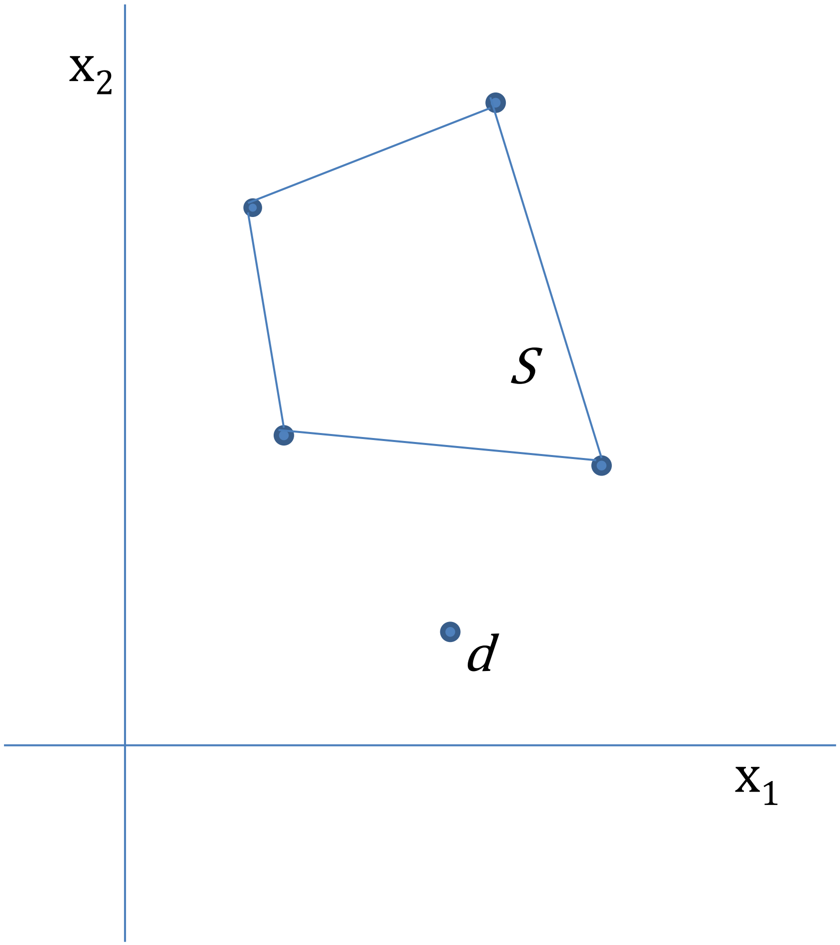
Thus we identify such a game with an ordered pair (S, d) where S is a subset of $\mathcal{R}^n$, and d is a point in $\mathcal{R}^n$ given by

$$d = (d_1, d_2, \ldots, d_n).$$

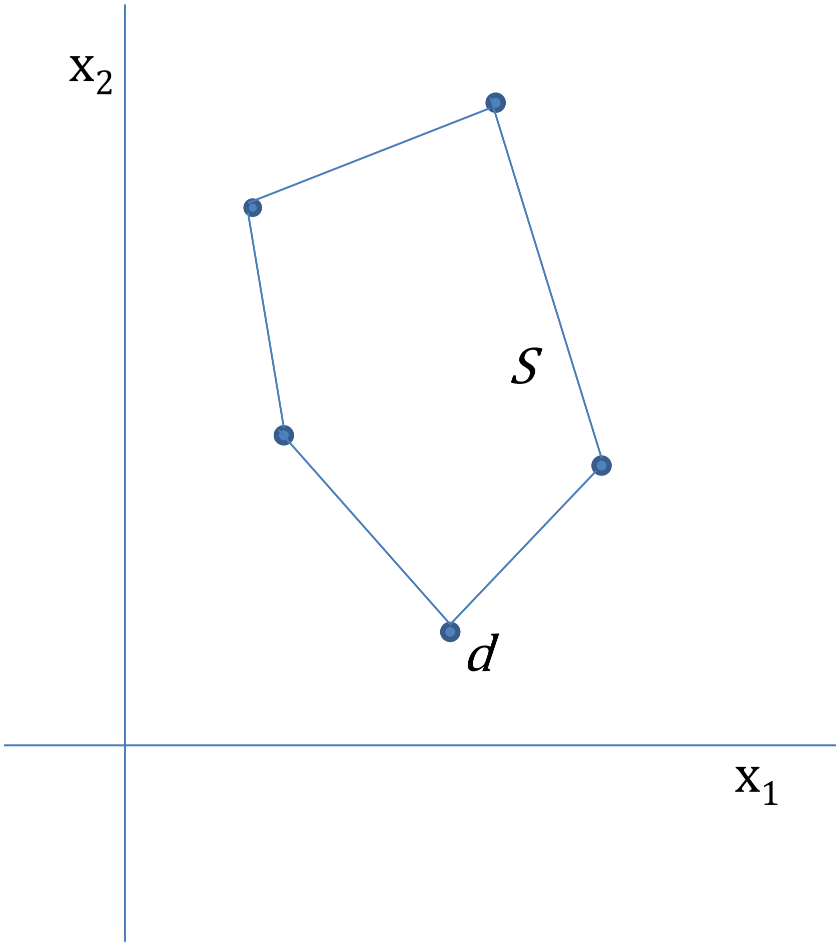
This represents the situation in which the players are trying to reach a unanimous agreement on the choice of payoff vector from S. If a unanimous agreement is not achieved, then the resulting payoff vector becomes d.


x_2
x_1


x_2
d
x_1

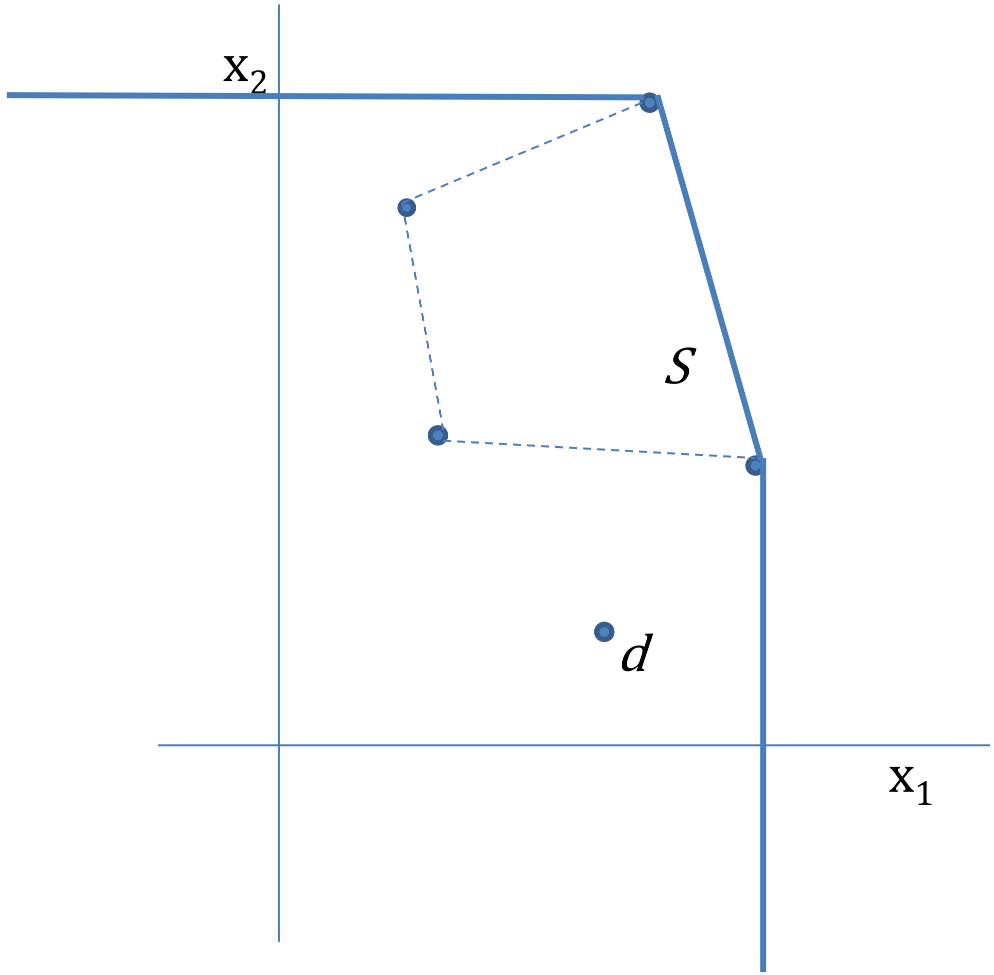


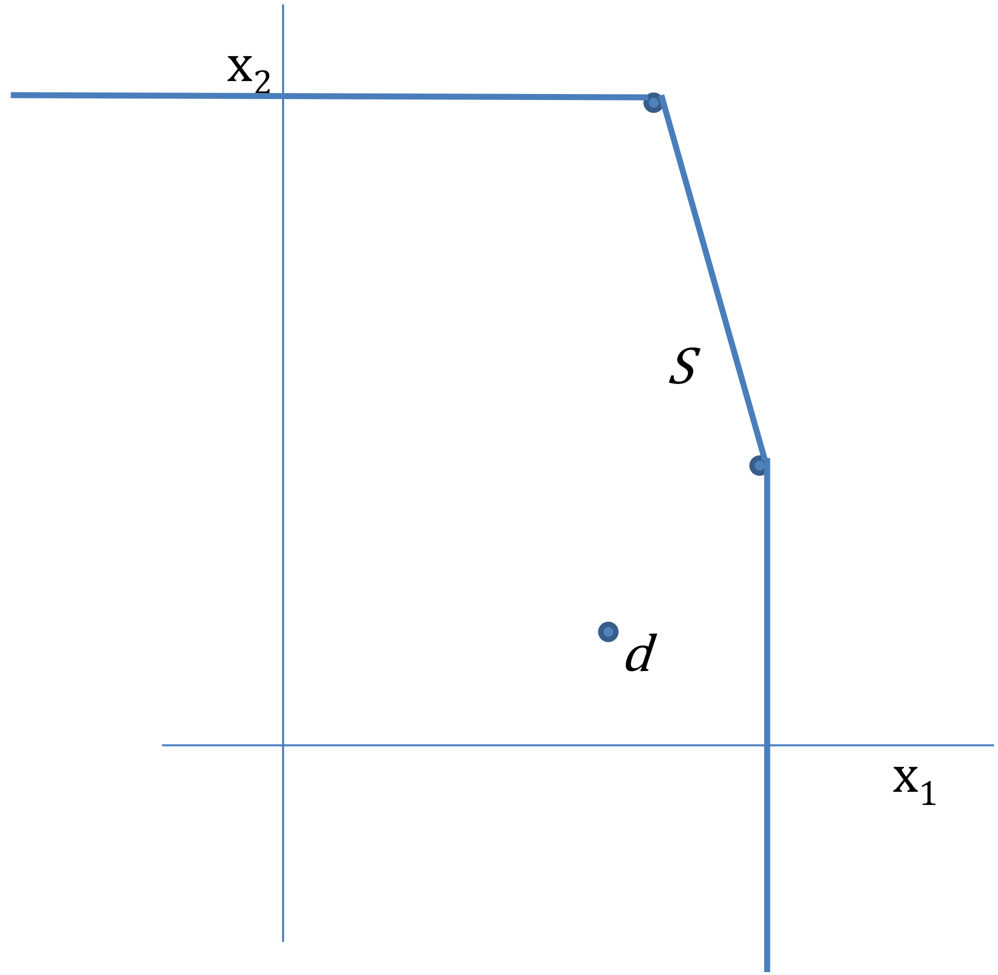
x2
S
d
x1



x_2
S
d
x_1

Individuals can freely dispose of utility




x2
S
d
x1

Bargaining problems

By an n-player *bargaining problem* we mean the problem of solving every game from a fixed nonempty set of n-player bargaining games.

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For example, in the classical two-player bargaining problem introduced by John Nash (1950), each game (S, d) is assumed to satisfy the following conditions:

(a) The set S is compact and convex.
(b) The point d belongs to S.
(c) There is a point $x \in S$ with $x > d$.

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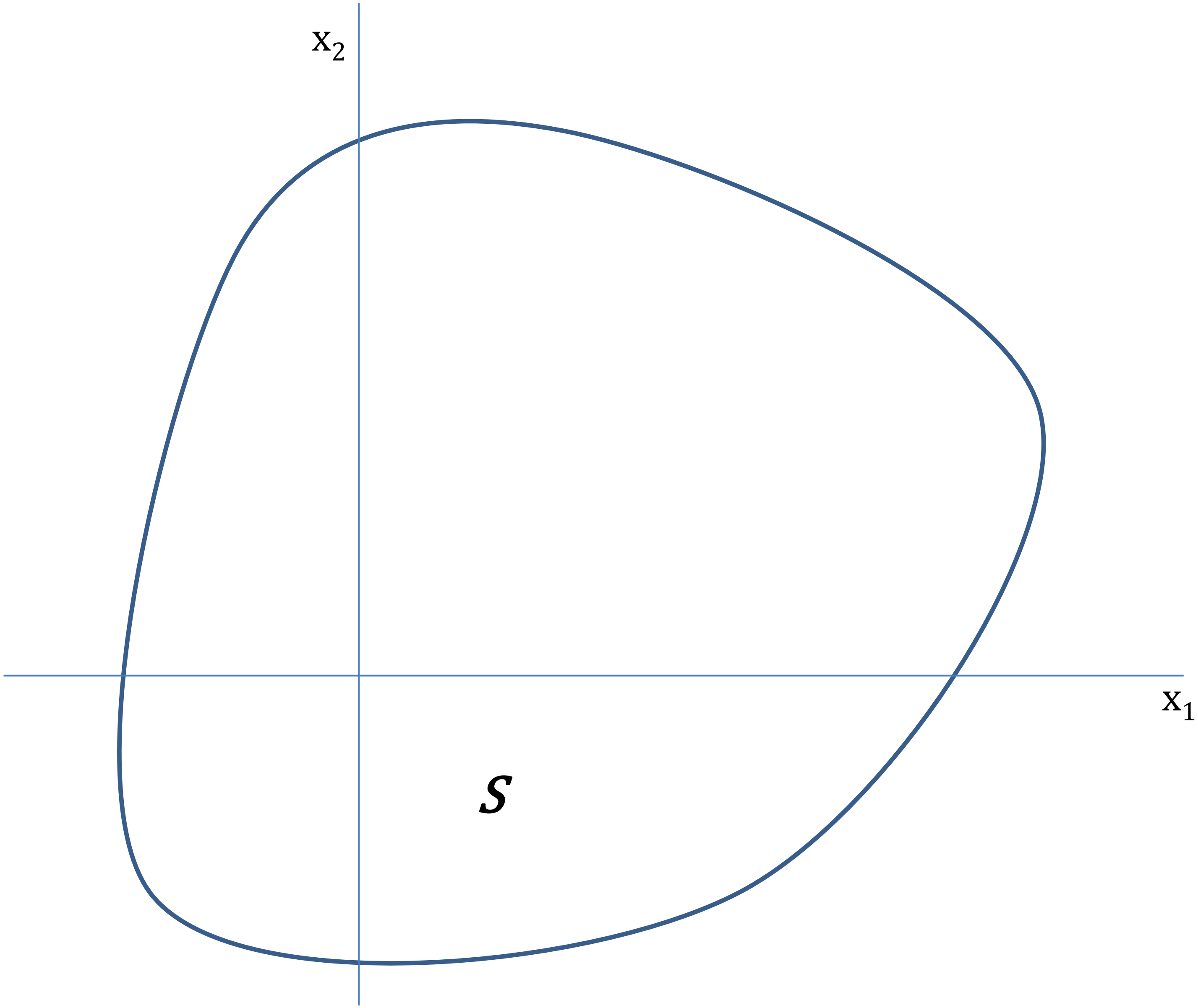
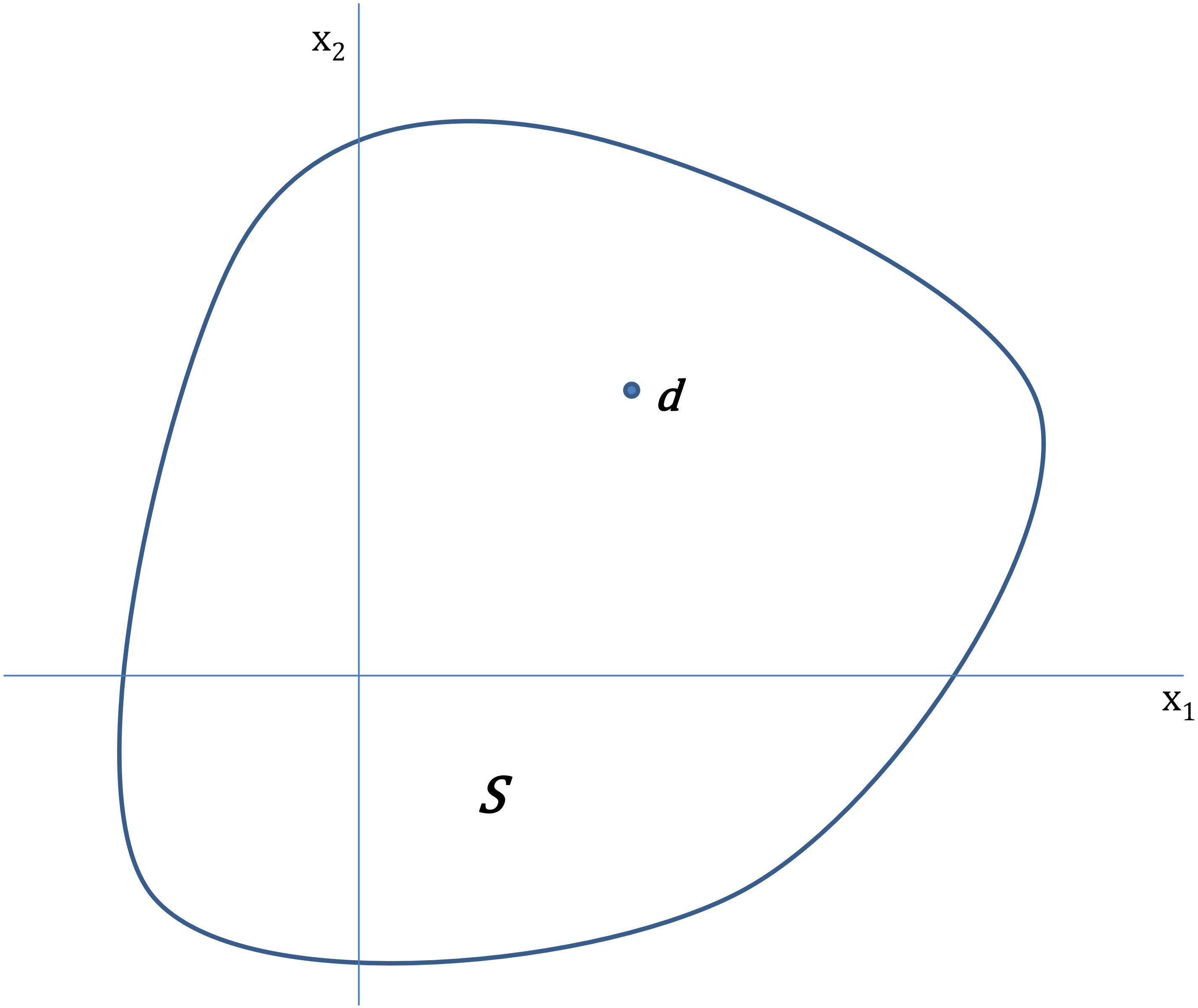
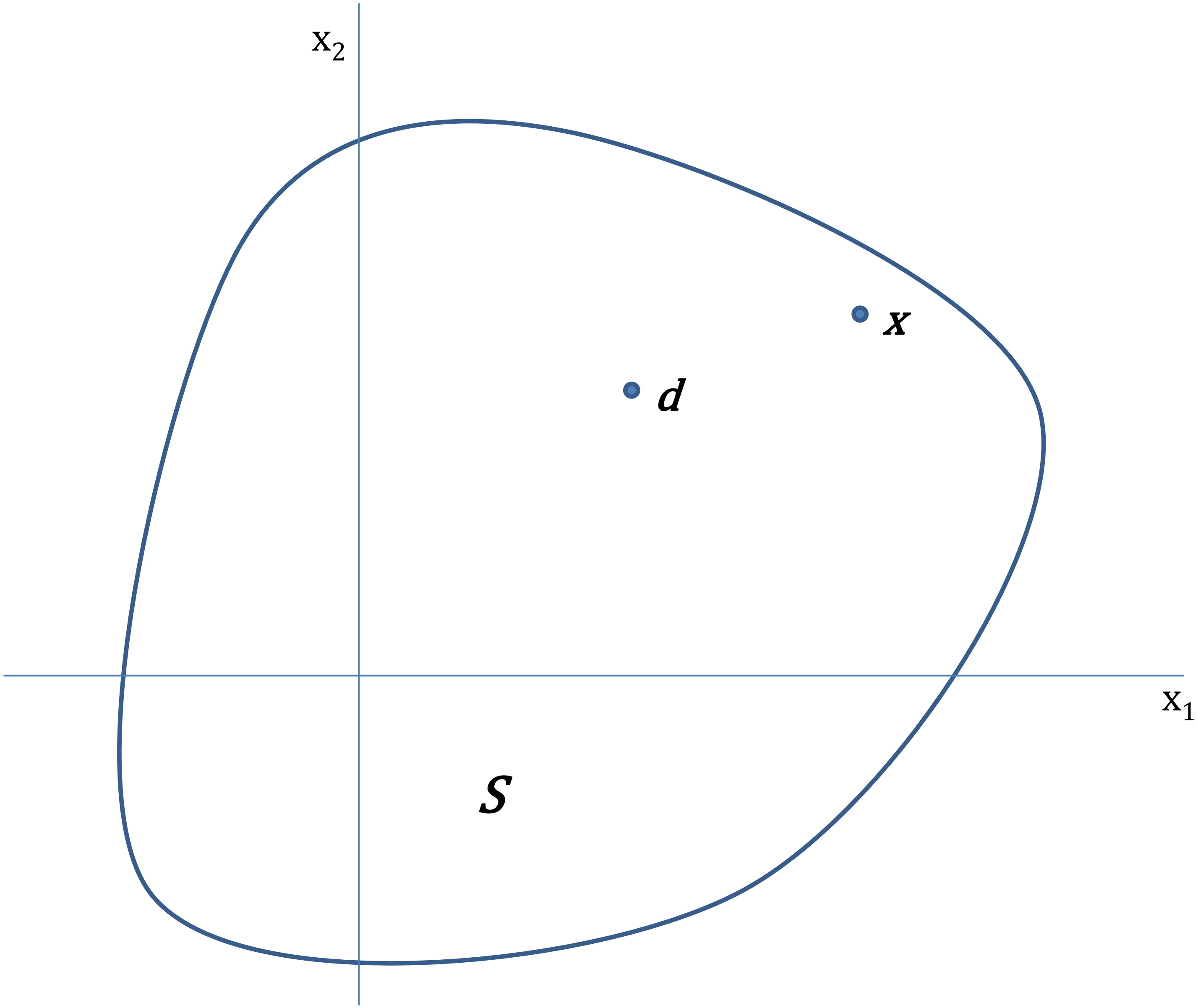
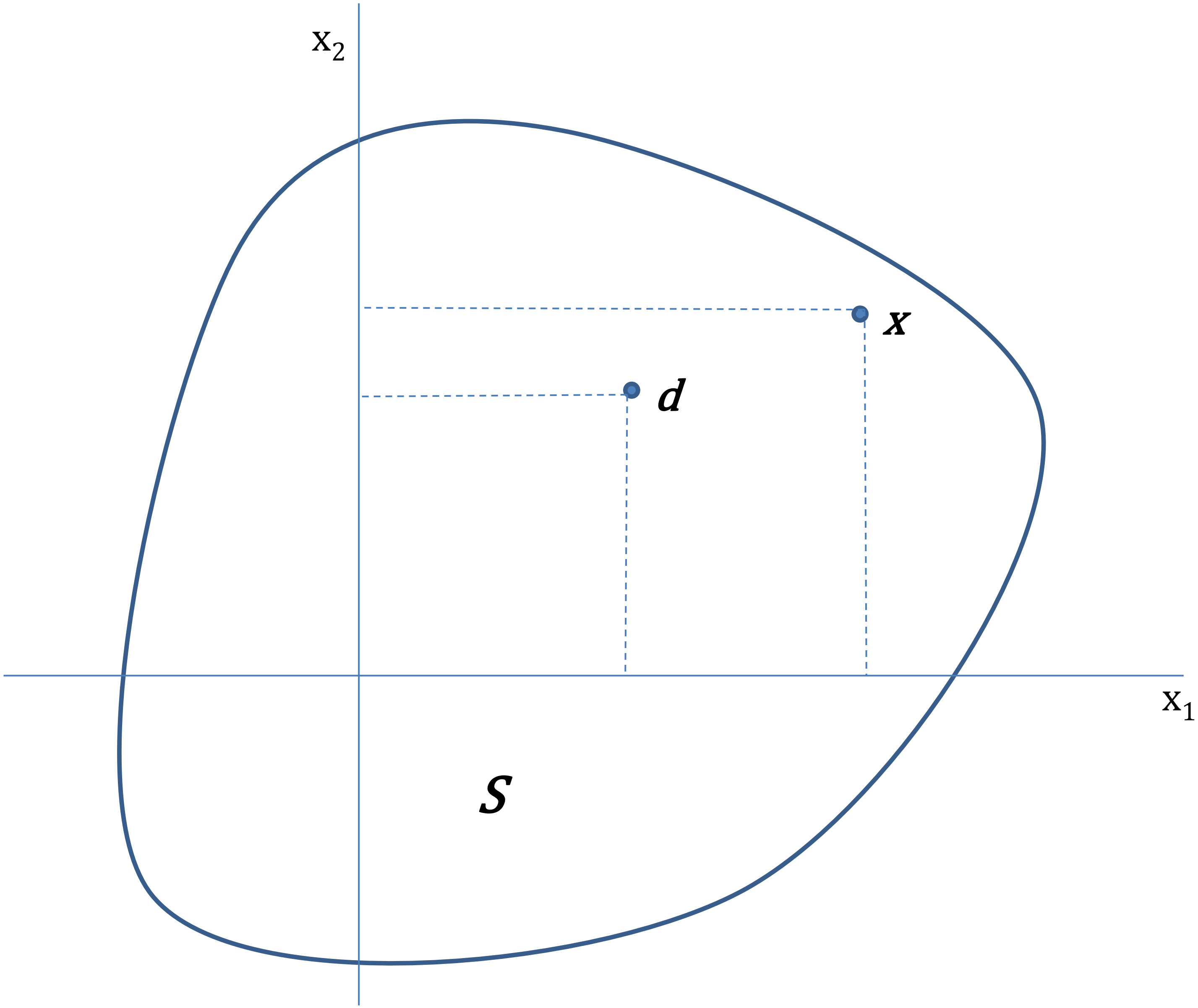
→ Bargaining can prove worthwhile to each player.


x_2
x_1
S

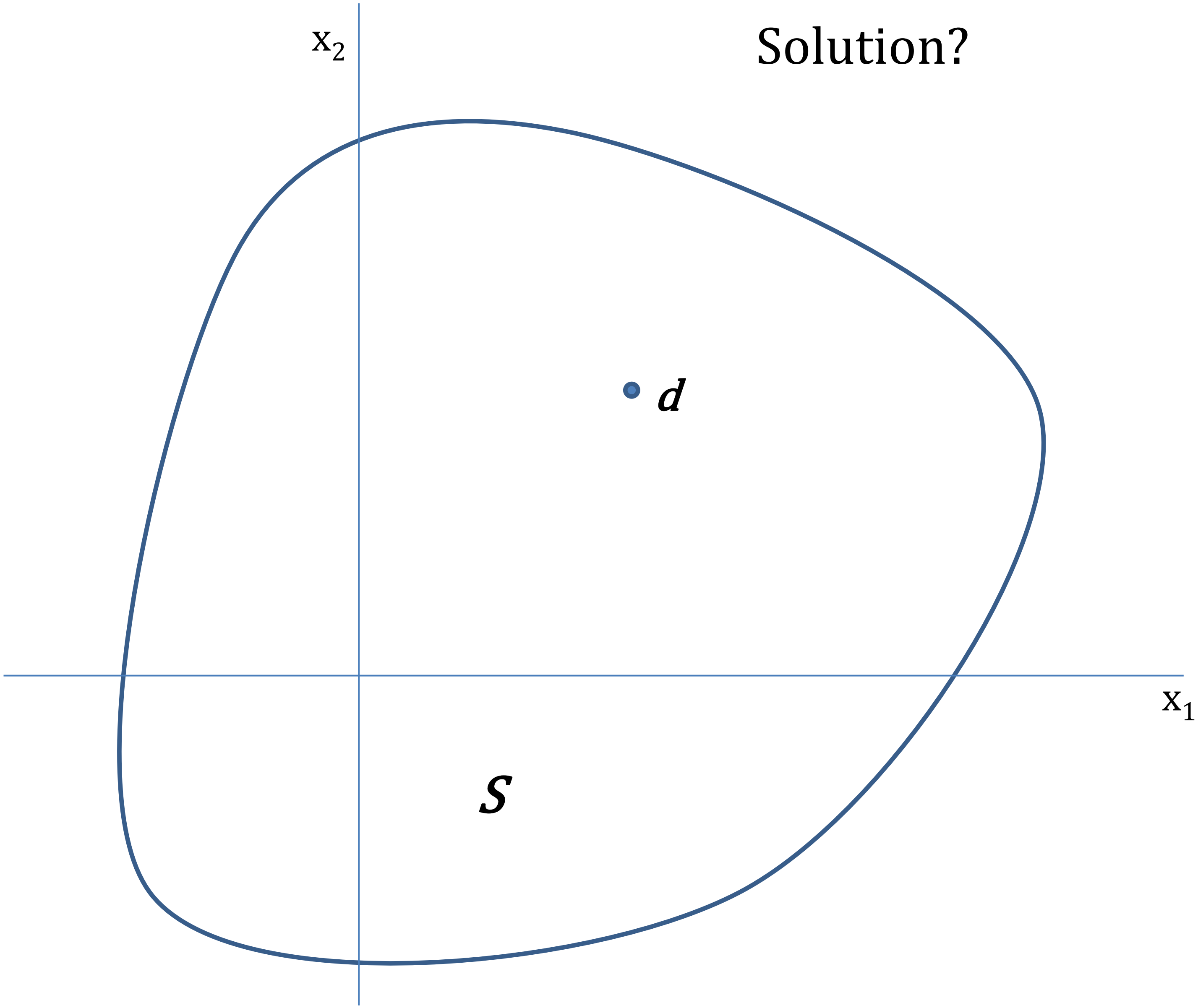


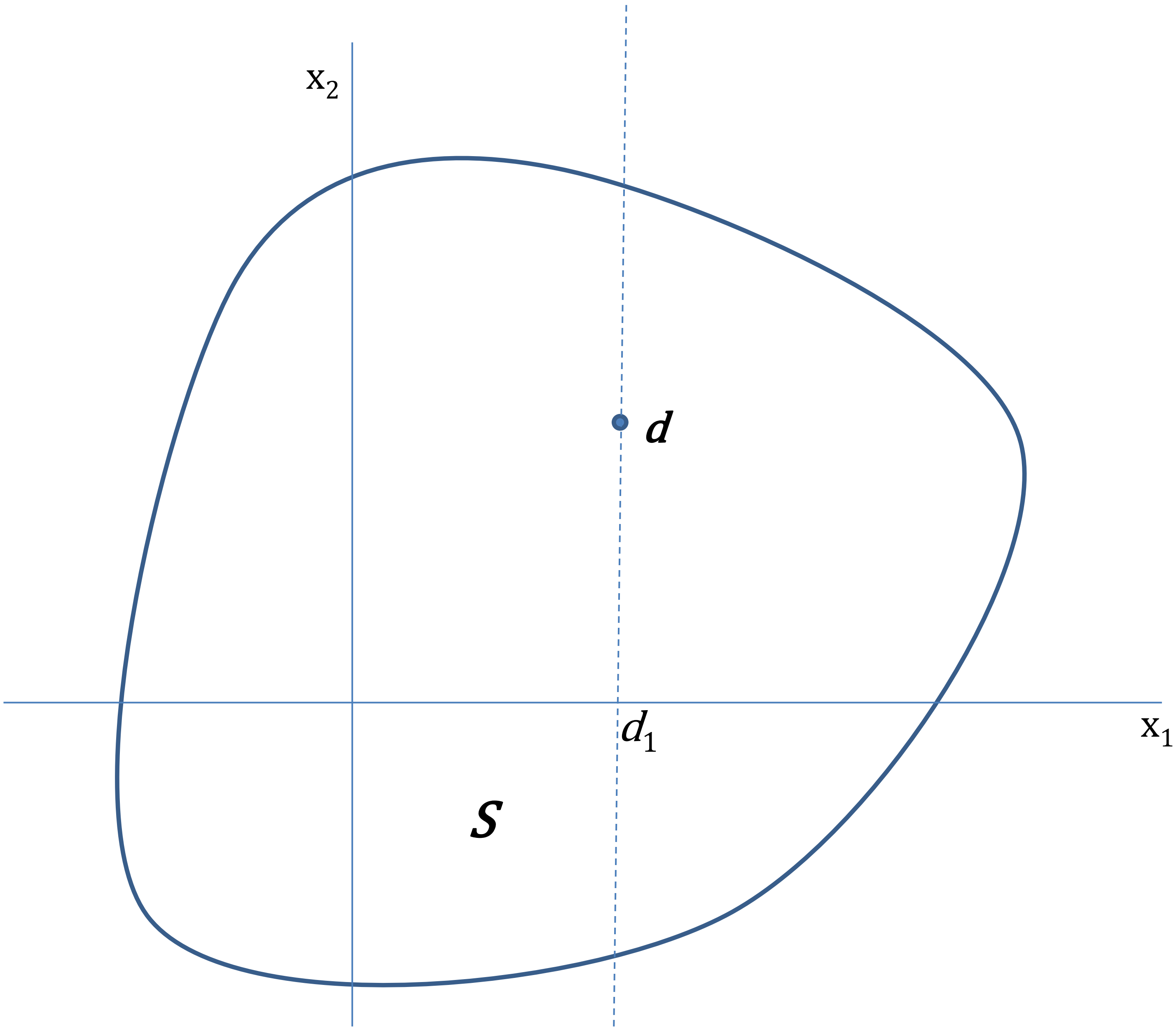
x_2
d
x_1
S


x2
x
d
x1
S

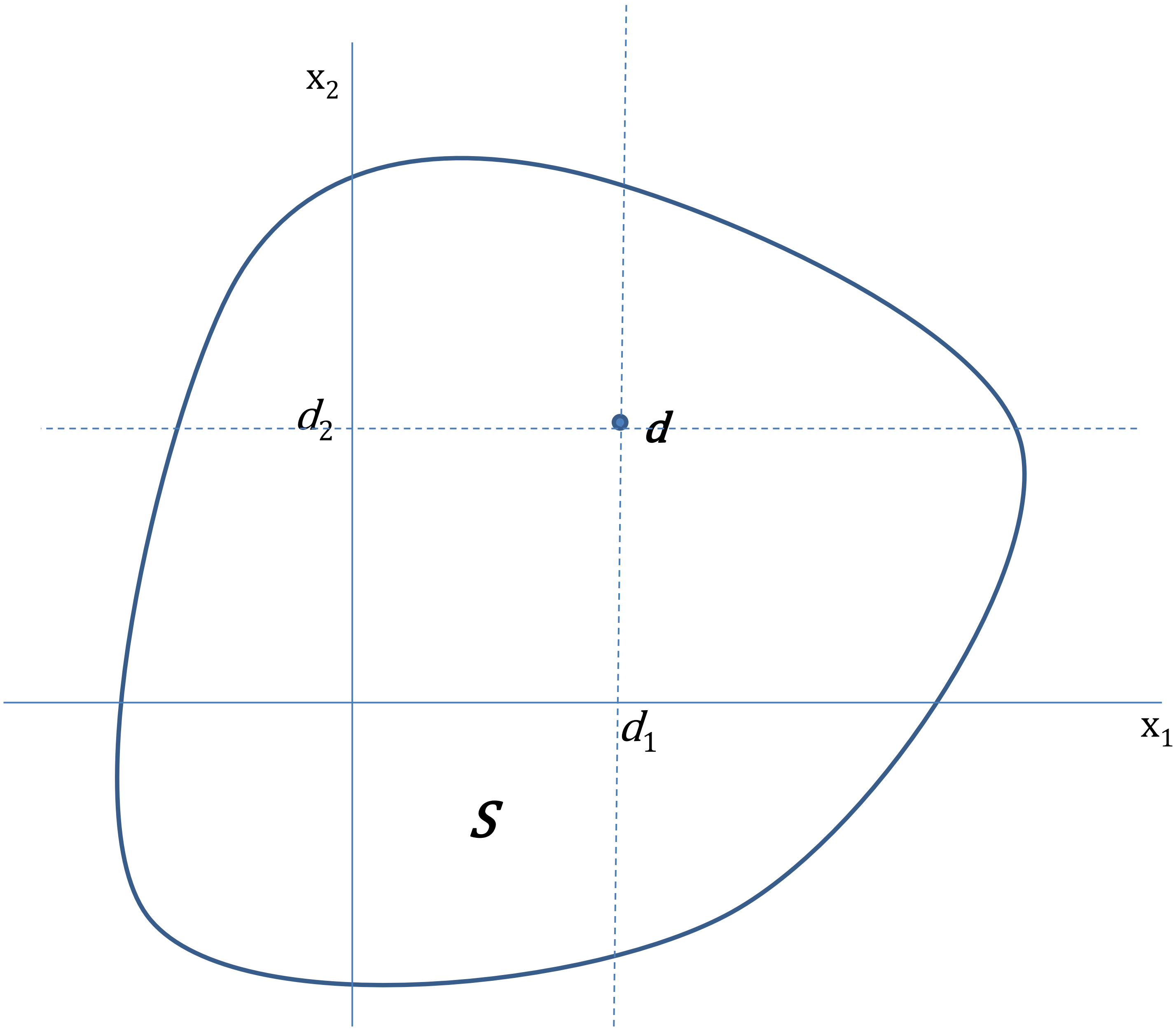

x_2
x
d
x_1
S

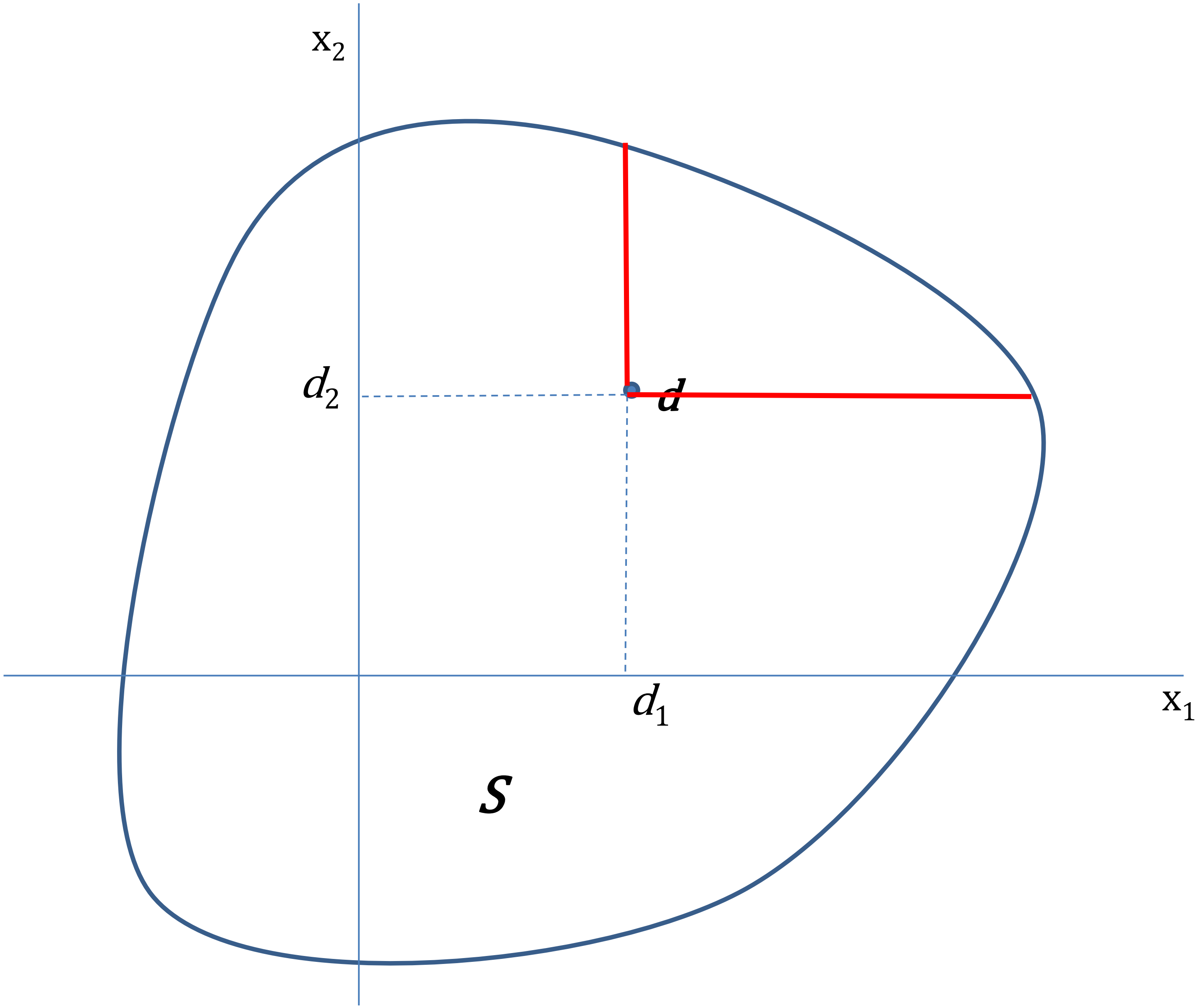
Solution?

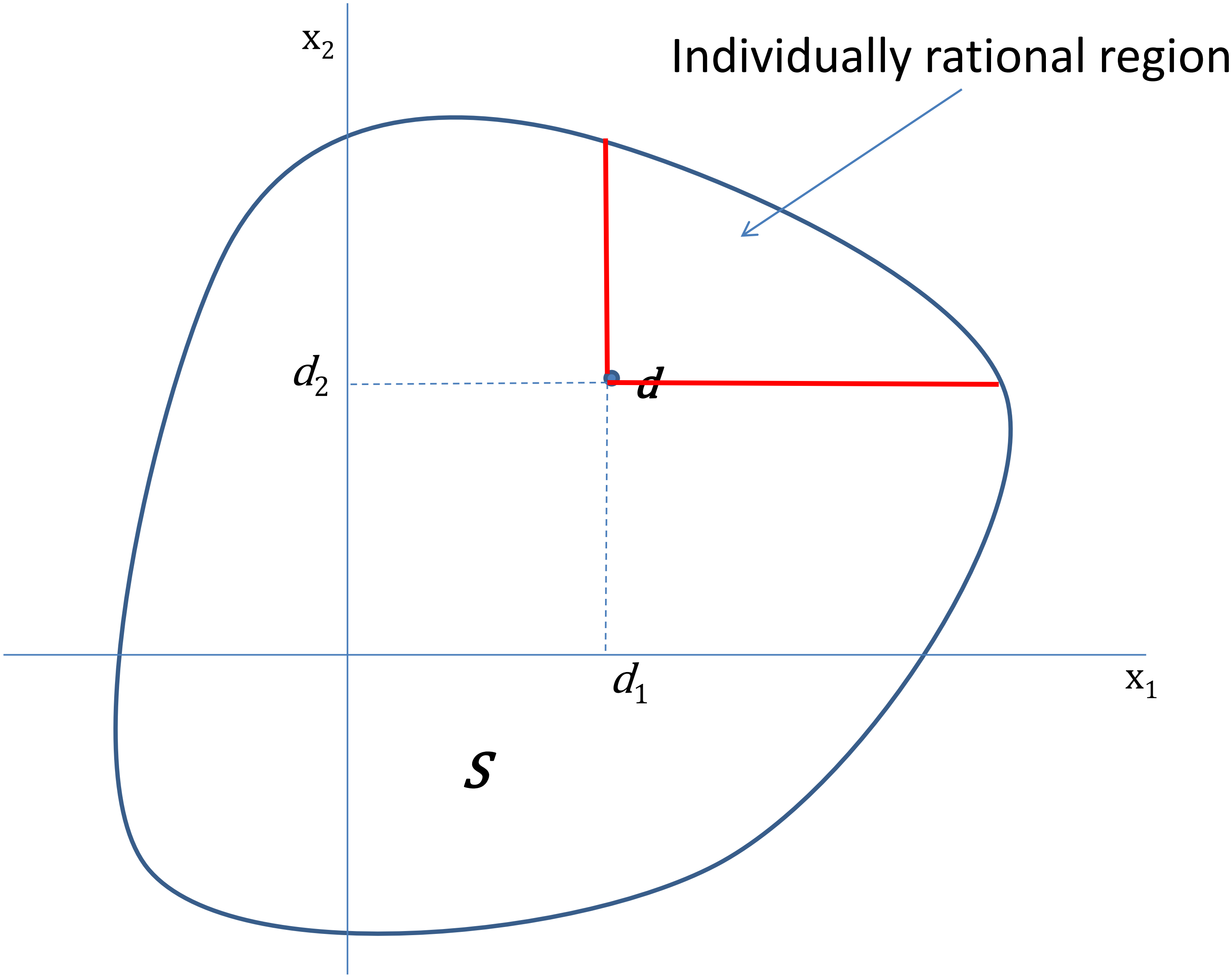




x2
d
d1
x1
S


x2
d2
d
d1
x1
S


x2
d2
d
d1
x1
S



Individually rational region
x_2
d_2
d
d_1
x_1
S

Solutions

Let $\mathcal{B}$ be an nonempty set of n-player bargaining games. A *solution* for $\mathcal{B}$ is a mapping f from $\mathcal{B}$ to the power set of $\mathcal{R}^n$ such that, for each instance (S, d) of $\mathcal{B}$, the value $f(S, d)$ is a nonempty subset of S.

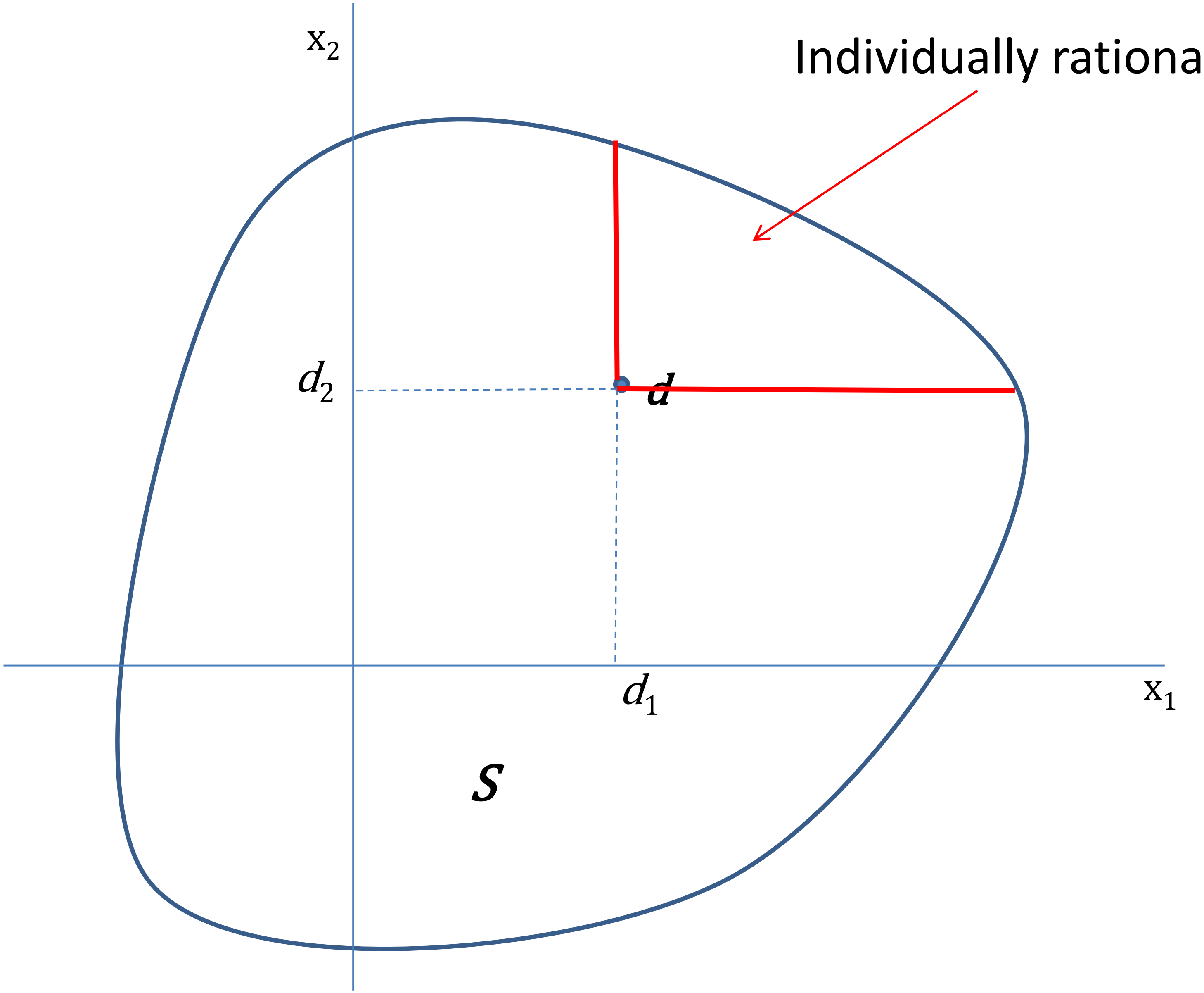
If f is a solution for a bargaining problem $\mathcal{B}$ and (S, d) belongs to $\mathcal{B}$, then the value $f(S, d)$ of f at (S, d) is called the f-*solution* of (S, d).

A solution is called *single point solution* if all its values are singletons.

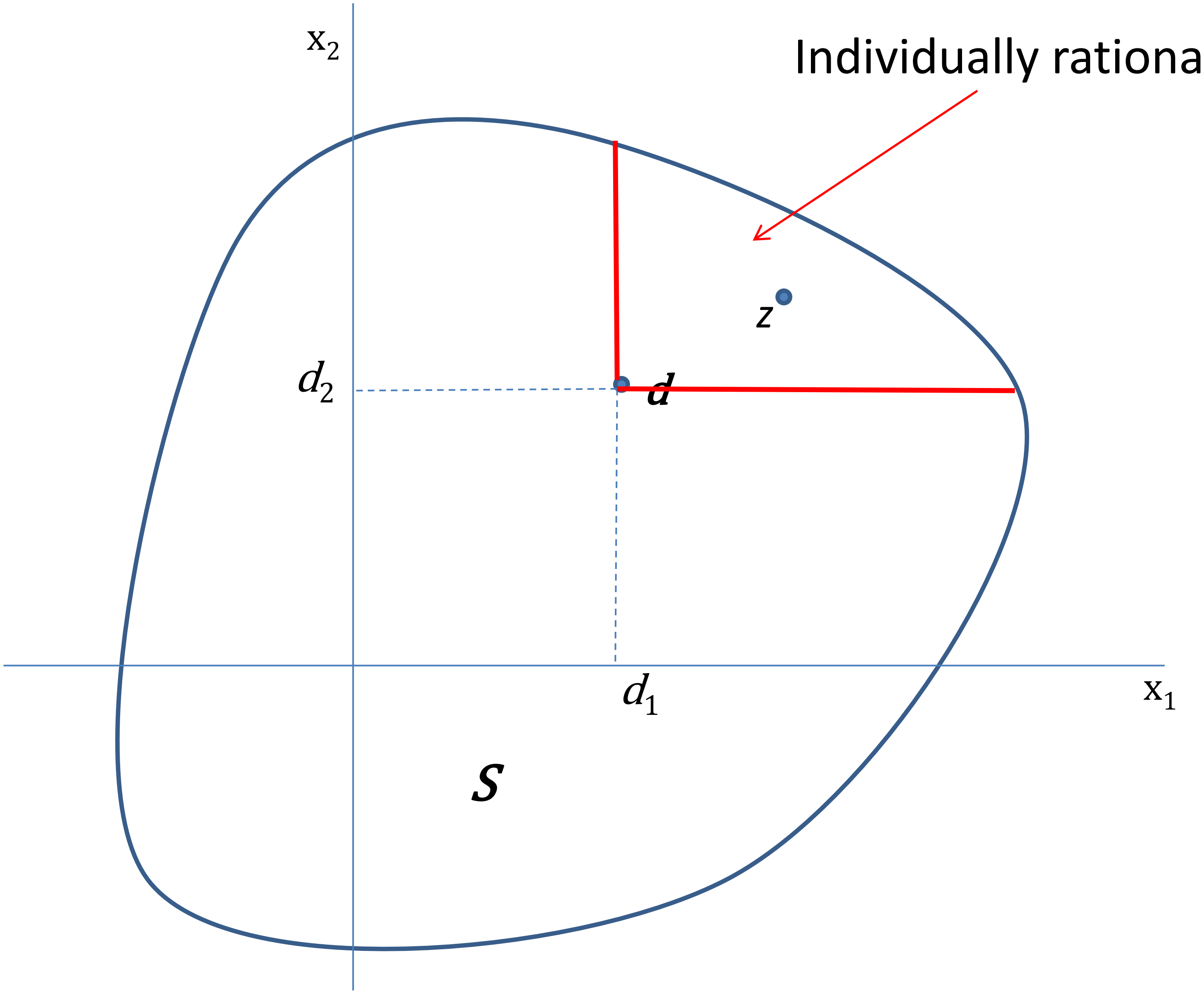
The Nash bargaining solution

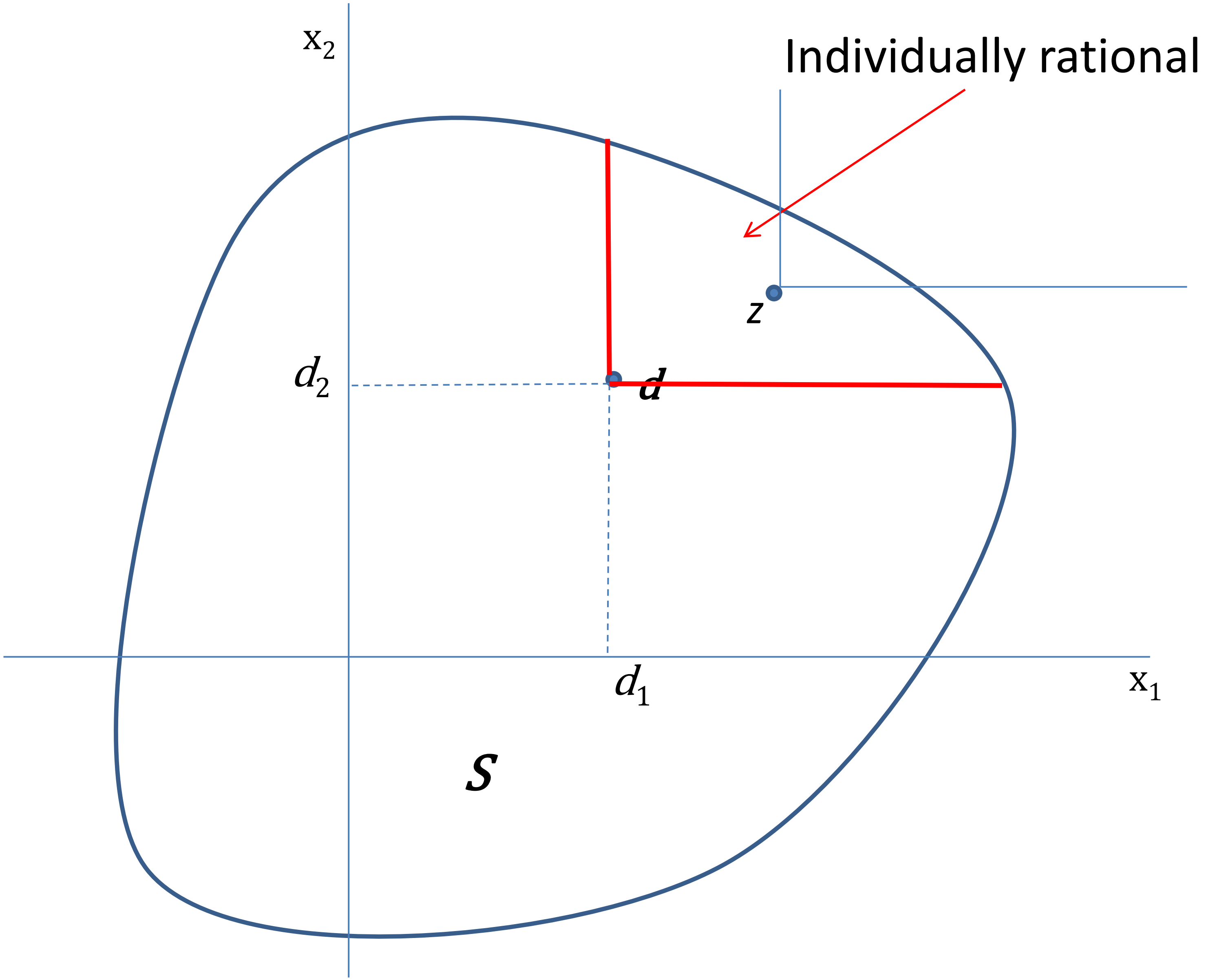
Without any doubt, the most famous single point solution is that of Nash. One of the attractive features of Nash's solution is that

- it is uniquely determined by a small number of simple properties,
- it can be computed by a simple numerical procedure,
- it can be supported by a game in extensive form.



Individually rational
x_2
d_2
d
d_1
x_1
S


Individually rational
x_2
z
d_2
d
d_1
x_1
S



Individually rational
x_2
z
d_2
d
d_1
x_1
S

1. *Pareto optimality.* For each instance (S, d) of $\mathcal{B}$, we have $(f(S, d) + \mathcal{R}^2_+) \cap S = f(S, d)$.

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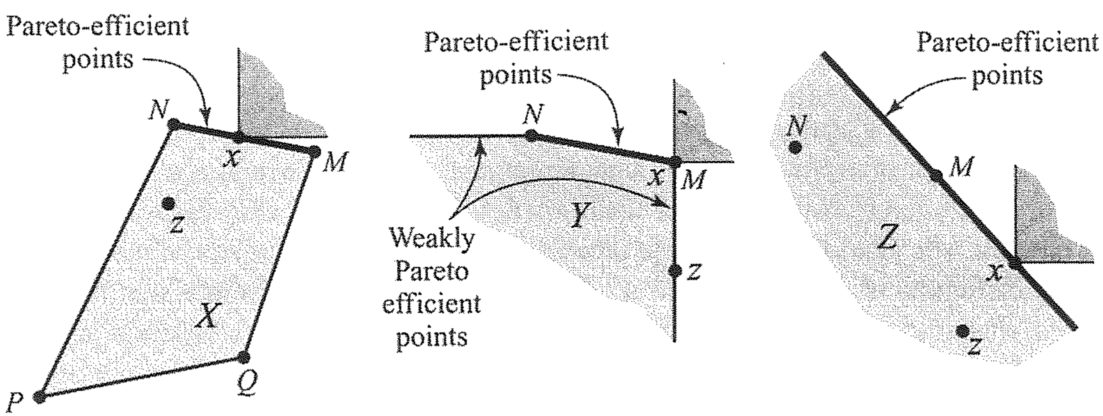
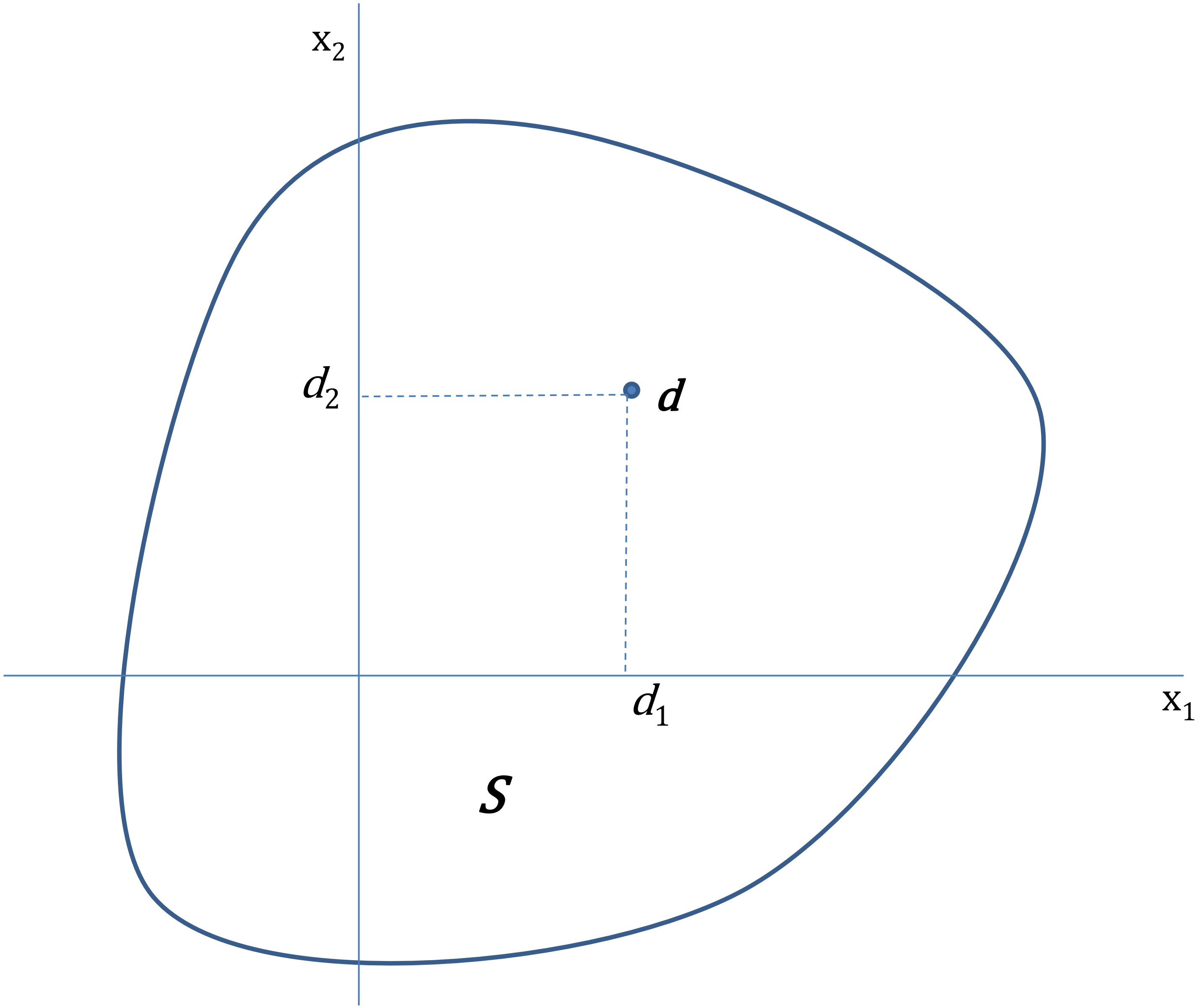
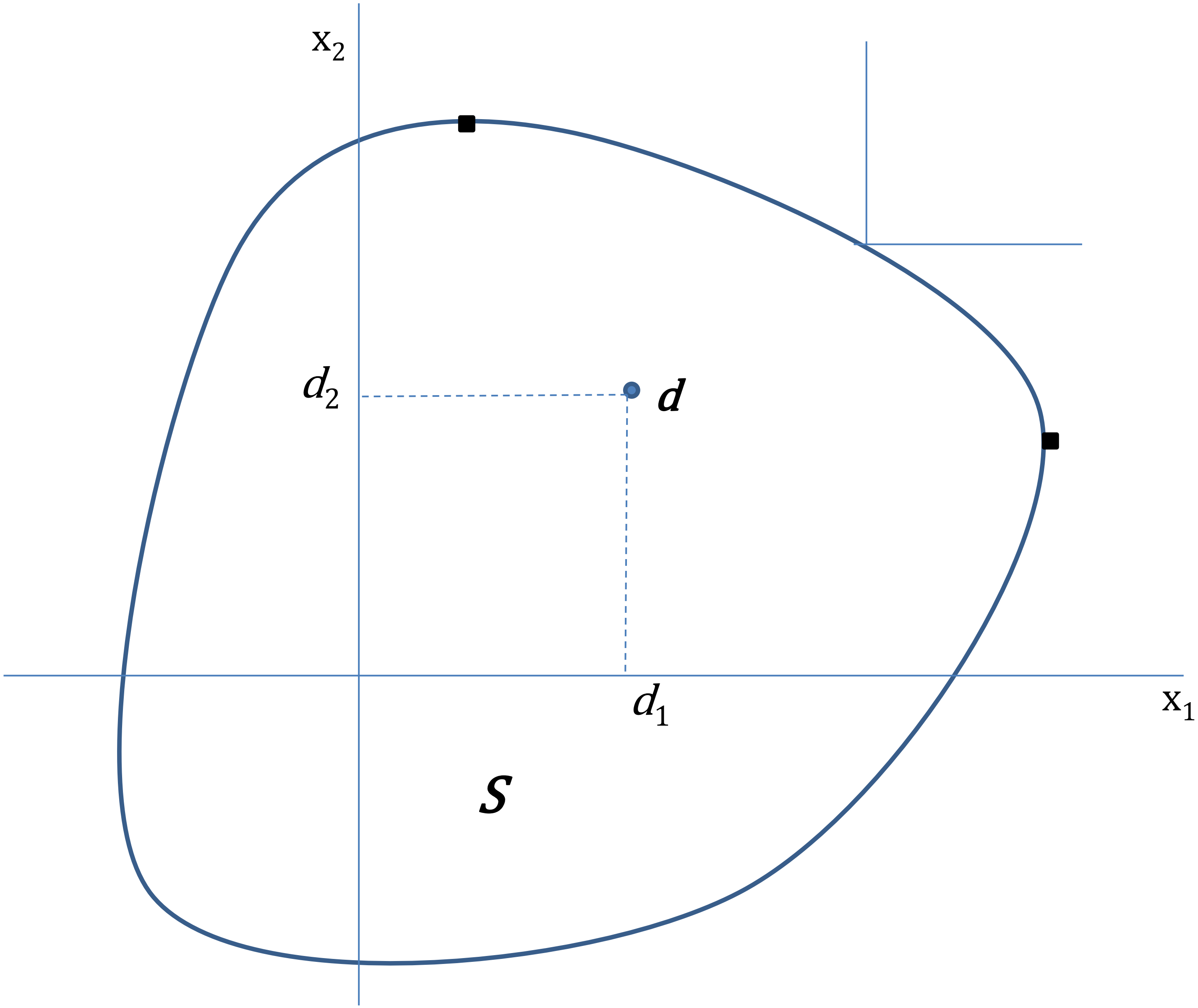
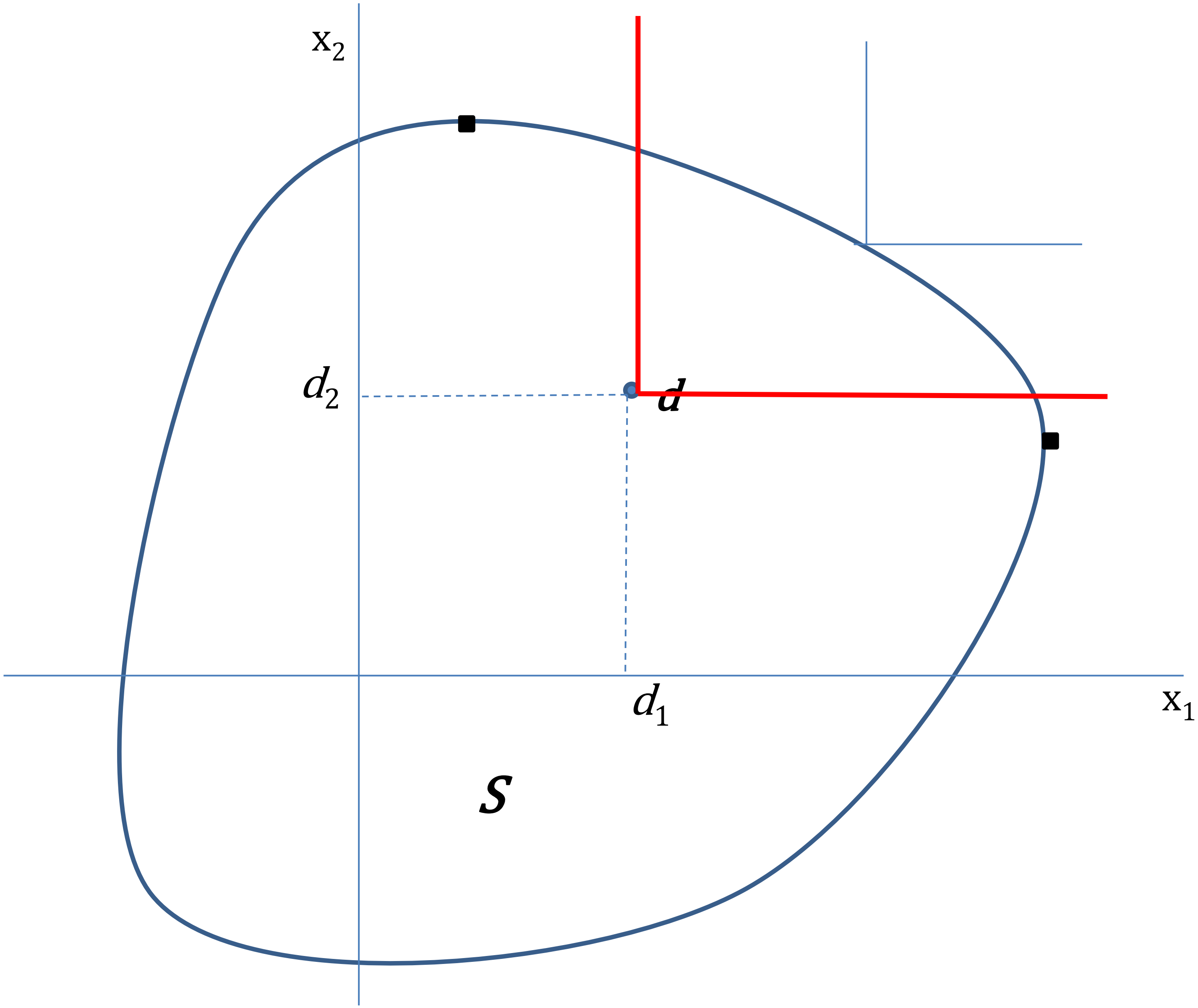


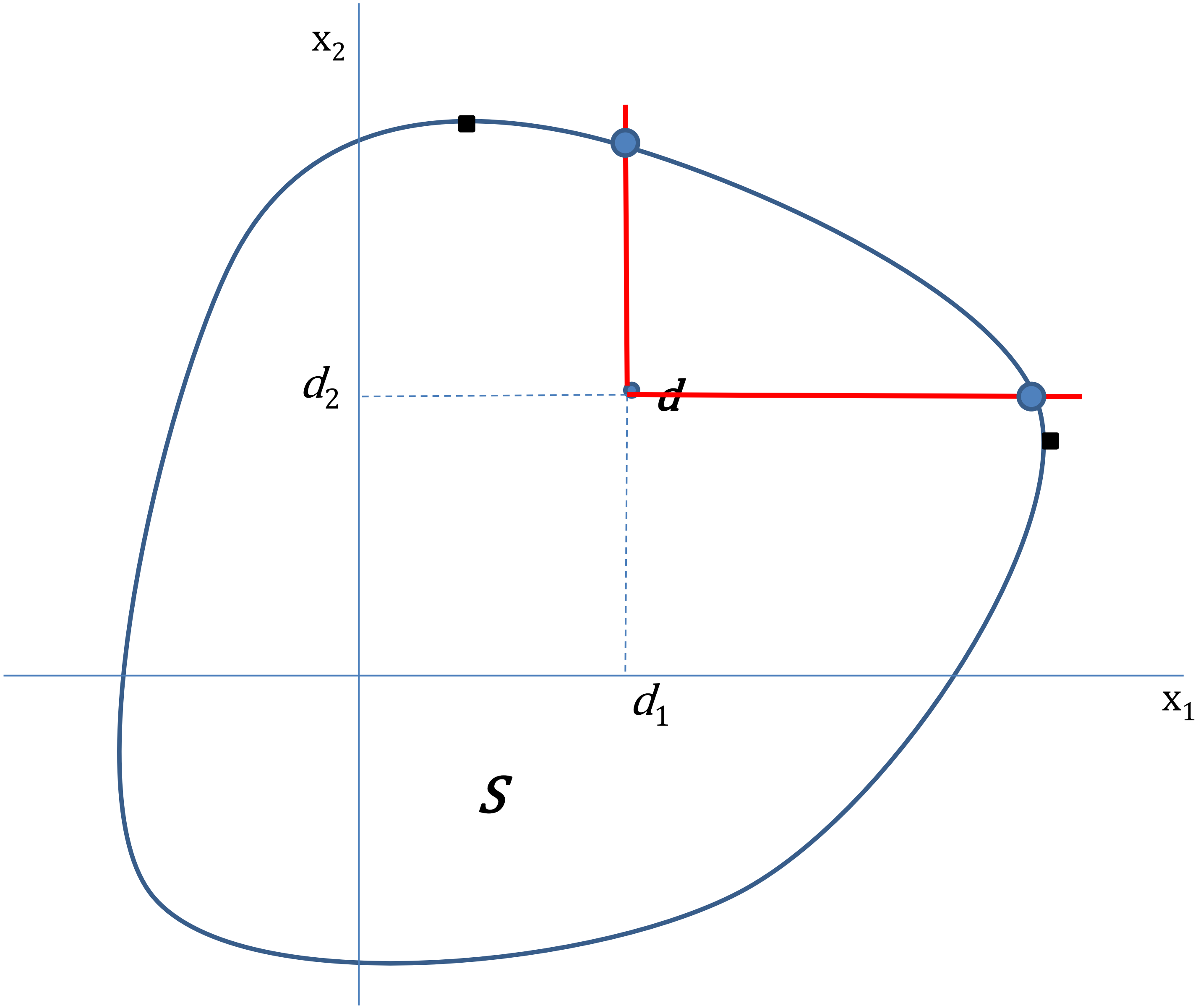


x_2
d_2
d
d_1
x_1
S

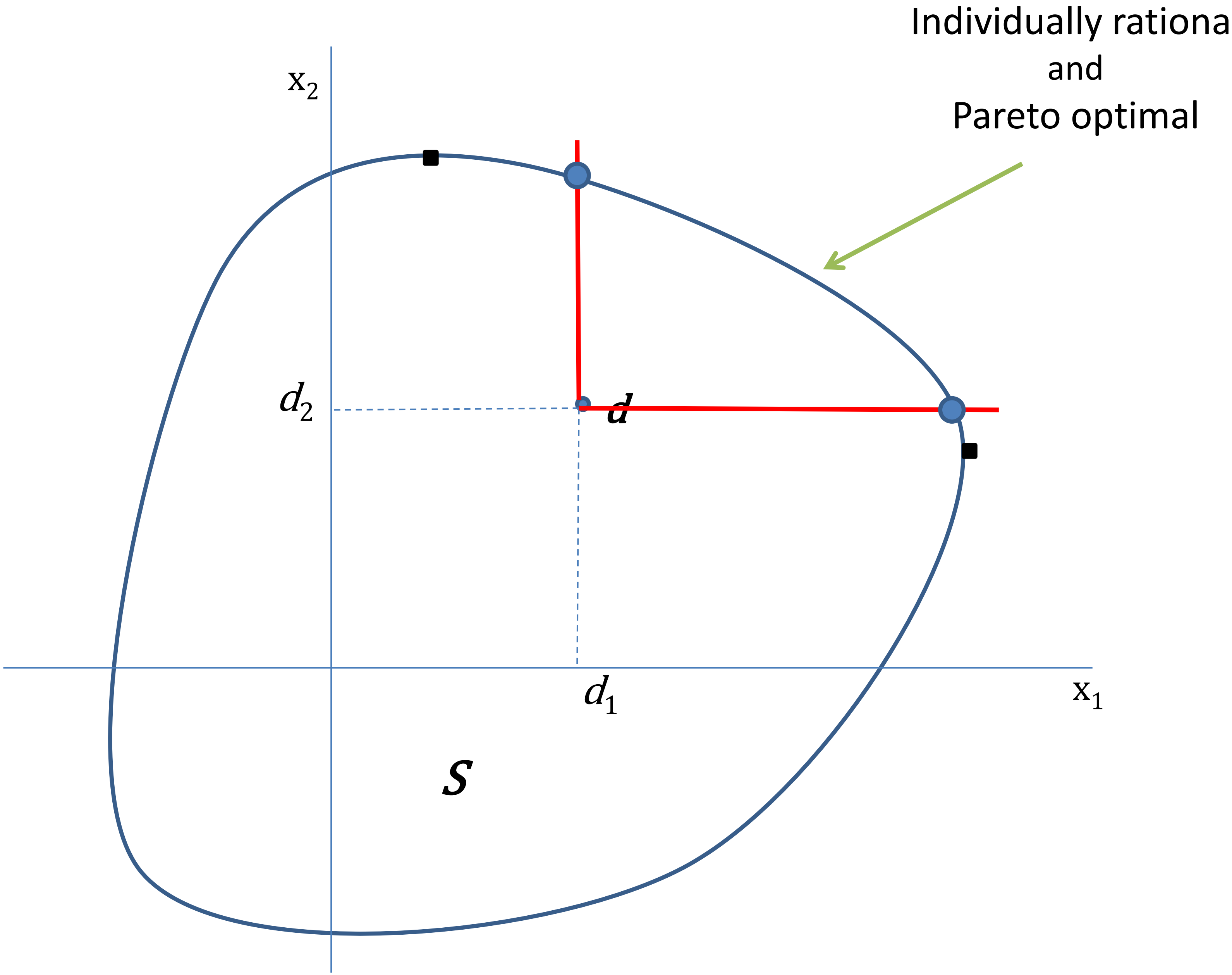

x2
d2
d
x1
d1
S



x_2
d_2
d
d_1
x_1
S



x_2
d_2
d
d_1
x_1
S



Individually rational
and
Pareto optimal
x2
d2
d
d1
x1
S

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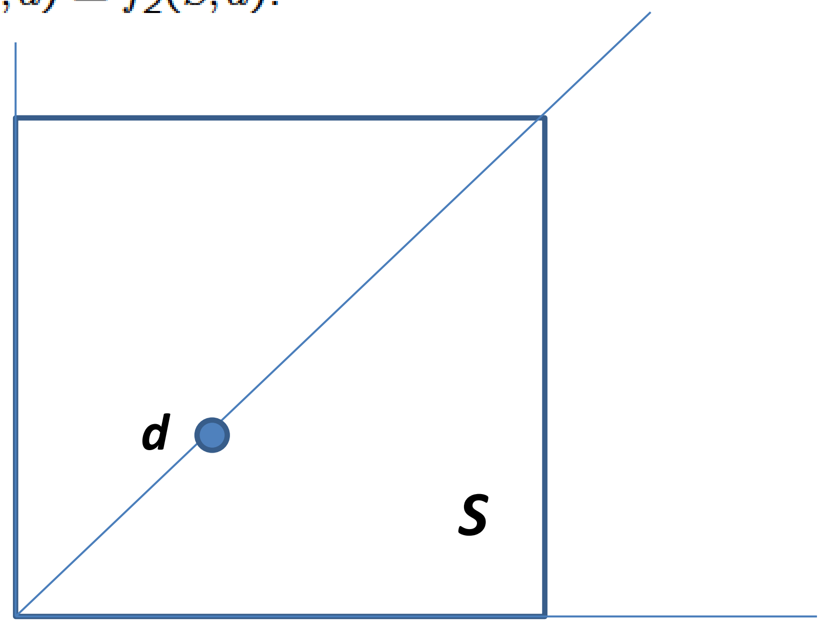

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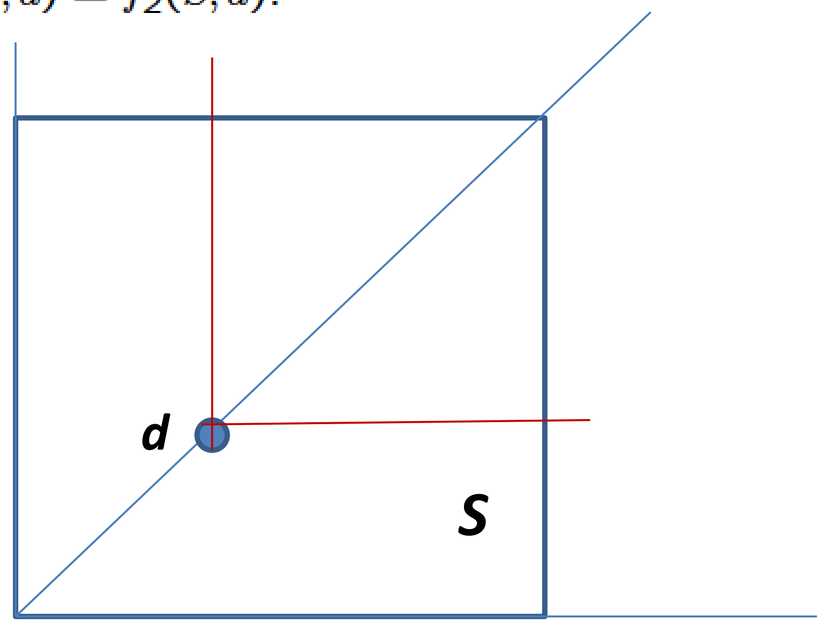

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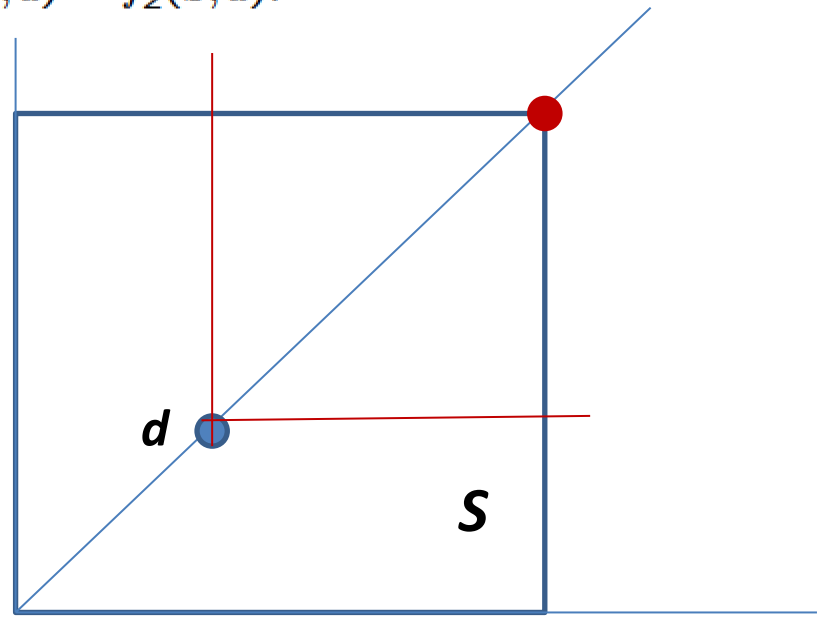

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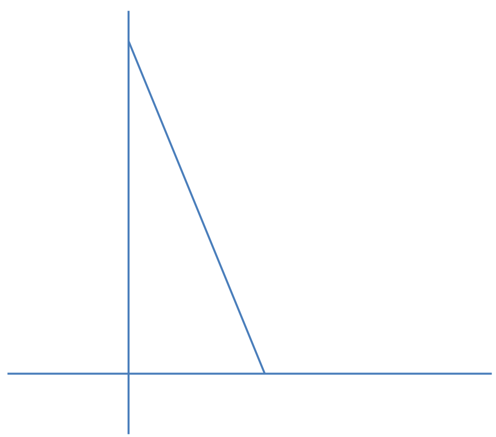

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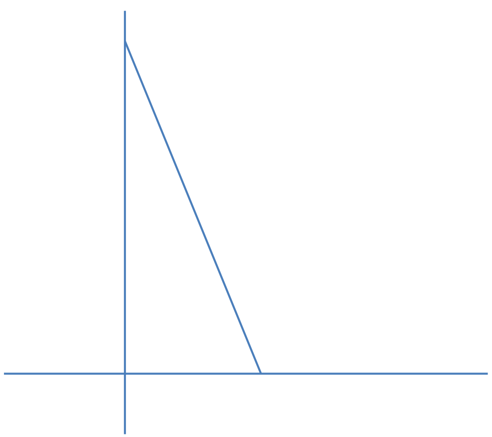

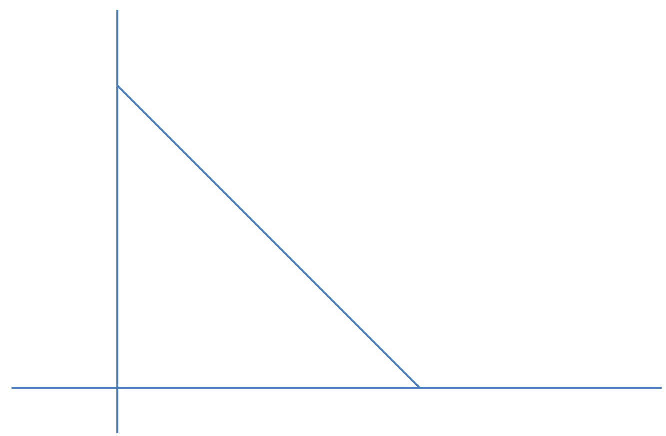

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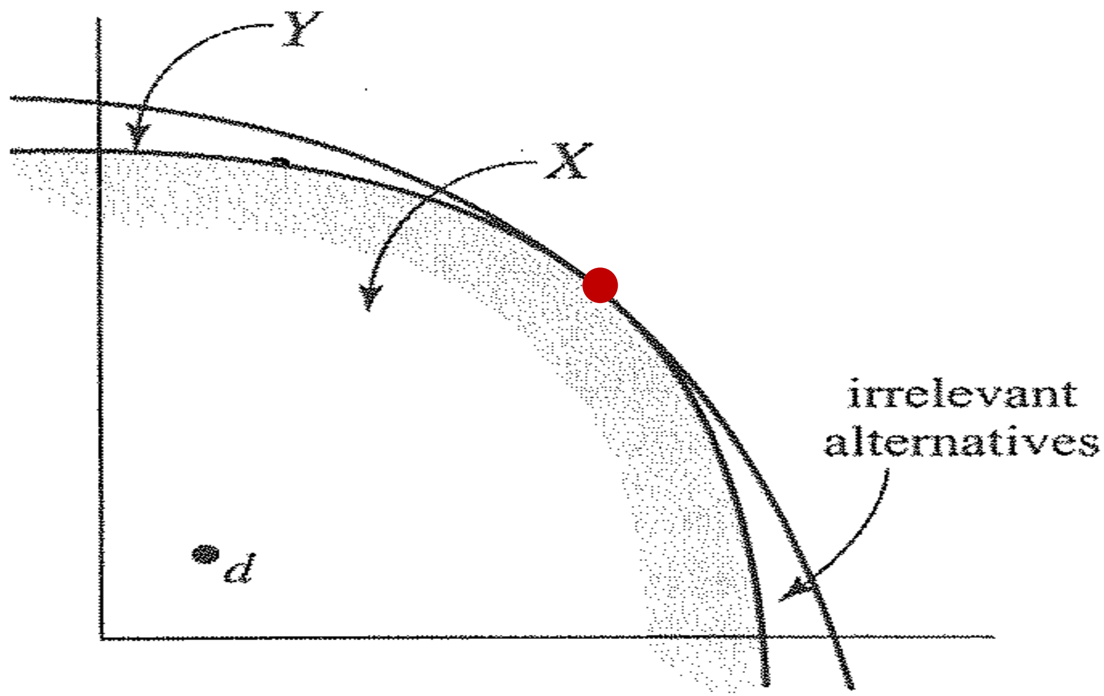

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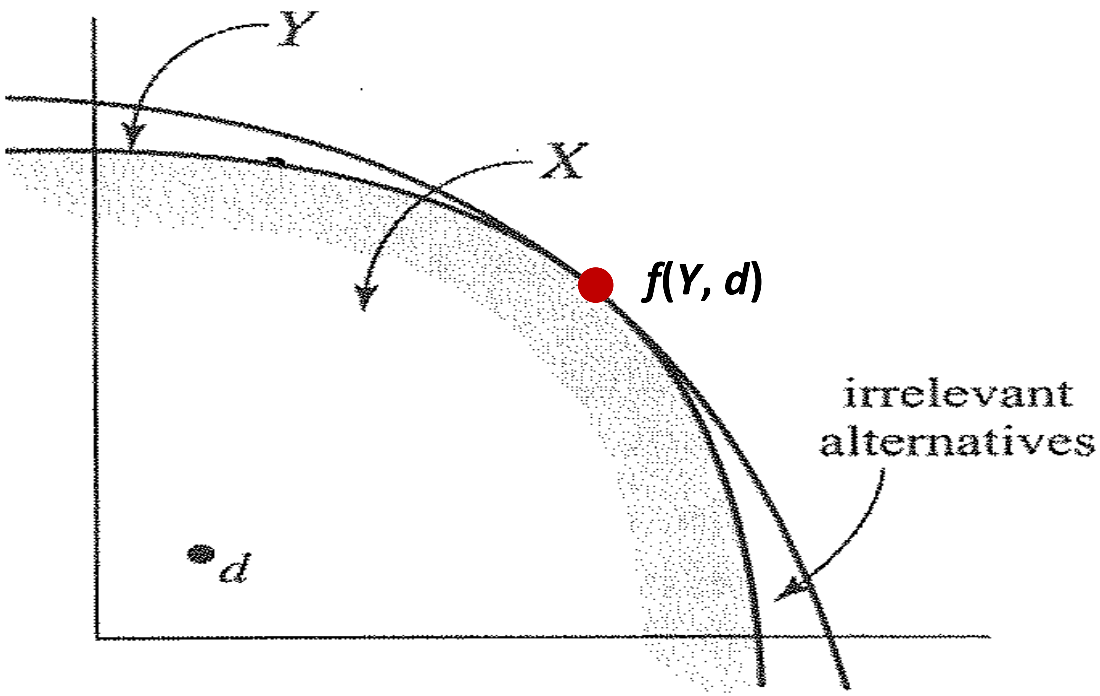

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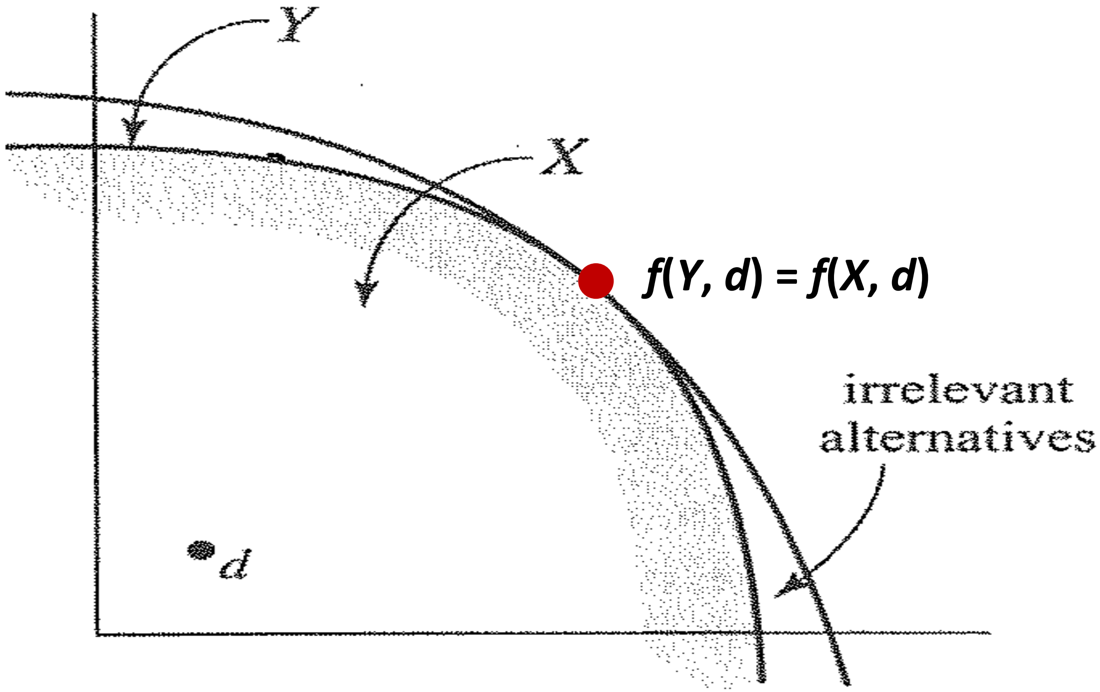

Nash proved that the only solution function on $\mathcal{B}$ satisfying these axioms is the function whose value at (S, d) is obtained by the maximization of the function

$$(x_1, x_2) \mapsto (x_1 - d_1)(x_2 - d_2)$$

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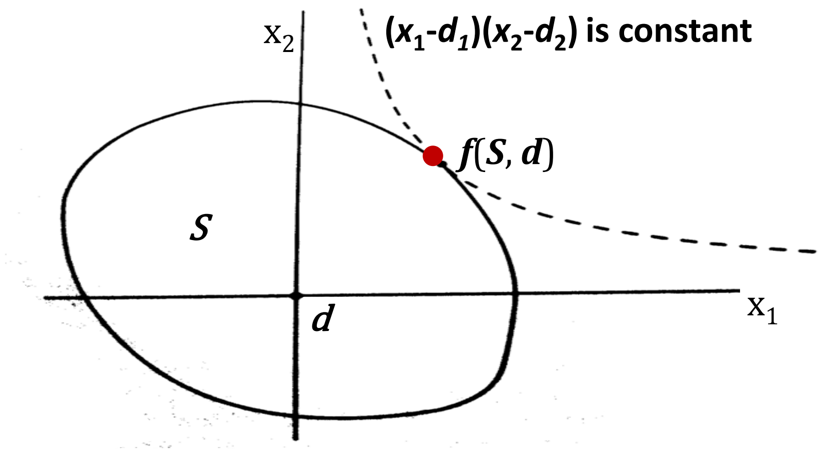

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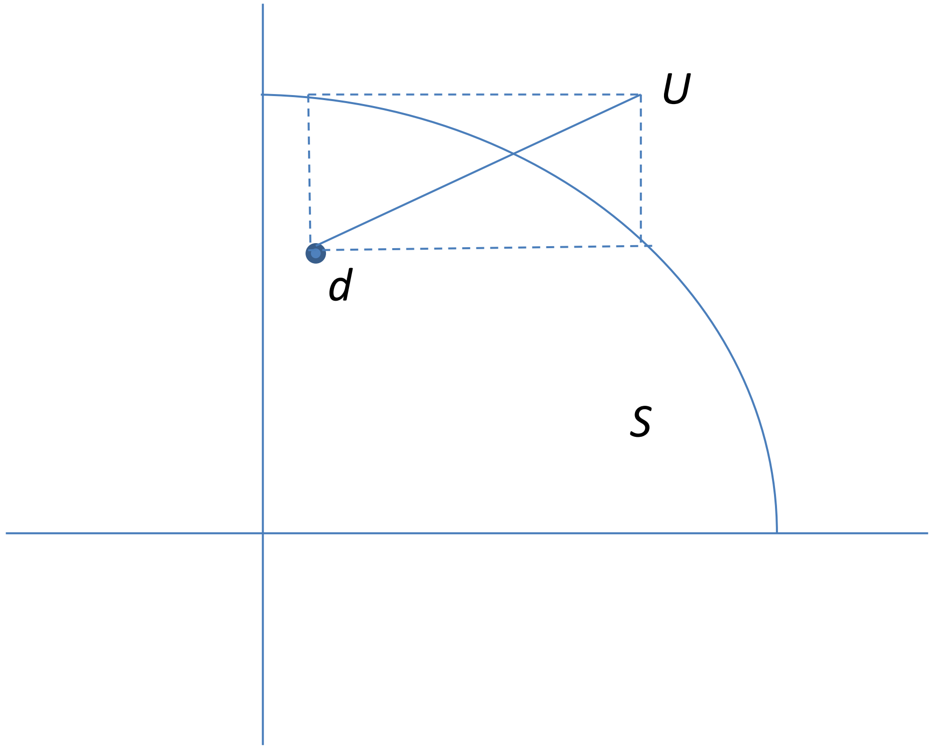
Alternative solutions

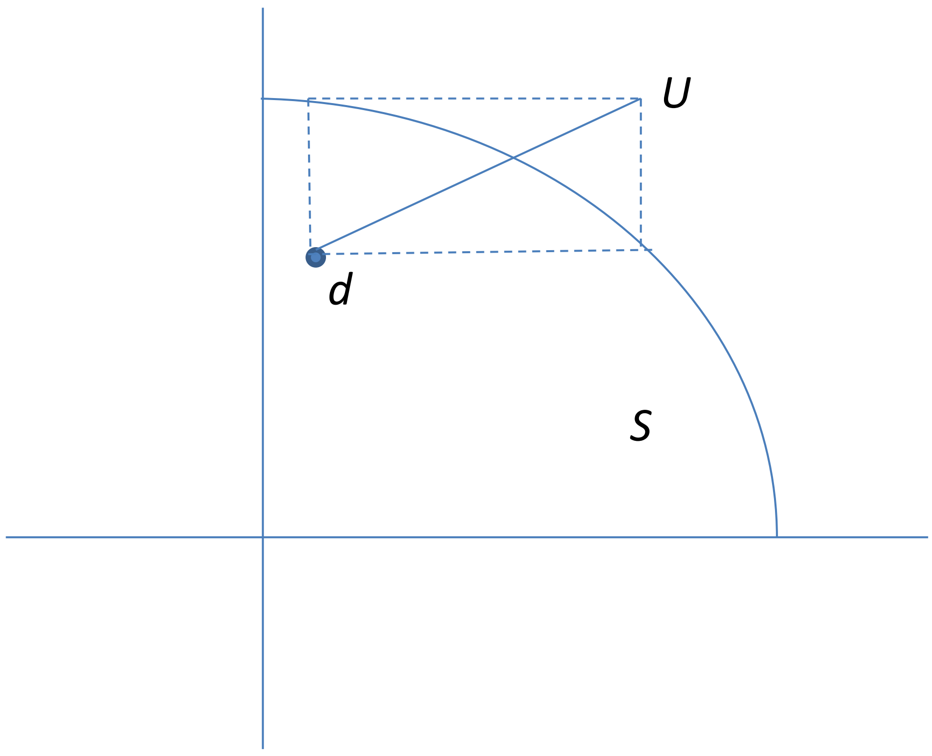
Instances of bargaining problems are relatively simple geometric objects. Thus a wide range of alternative solution concepts, based on elementary geometric operations, have been proposed.

- The Kalai-Smorodinsky solution; (Kalai and Smorodinski, 1975)
- The Discrete Raiffa solution; (Raiffa, 1951, 1953)
- The Shapley-Shubik solution; (Shapley, 1969; Shubik, 1982)

Kalai-Smorodinski Solution (1975)



(Raiffa)-Kalai-Smorodinski Solution (1975)



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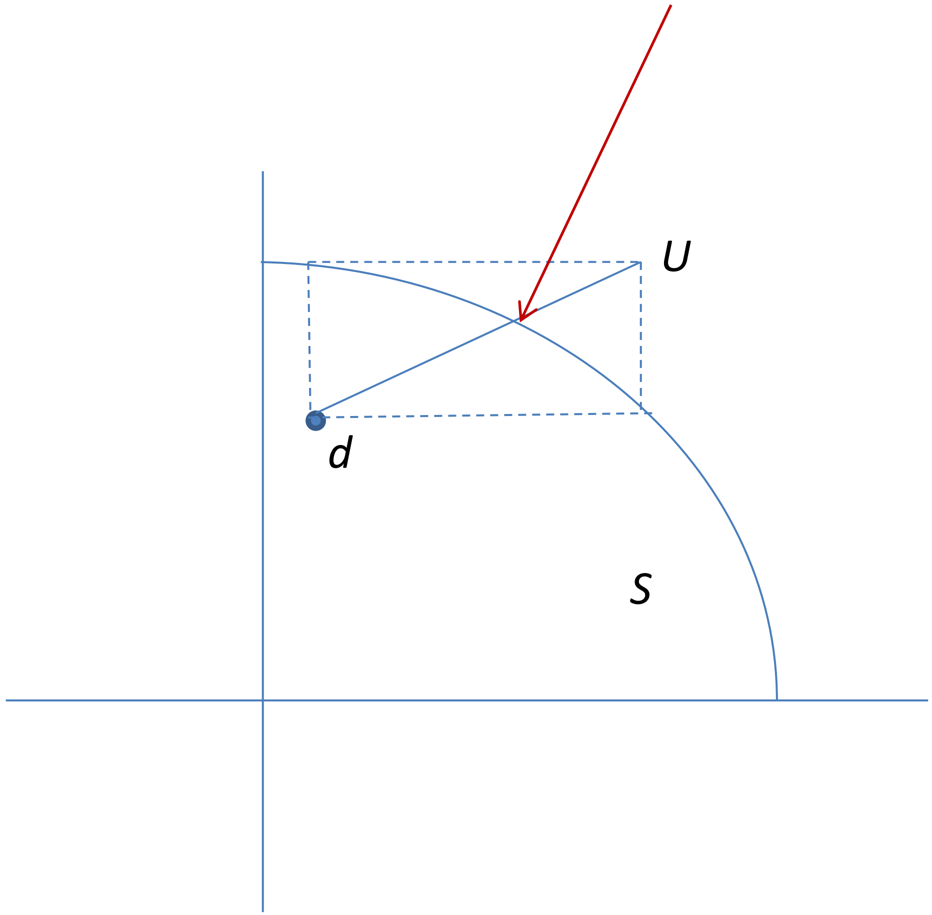

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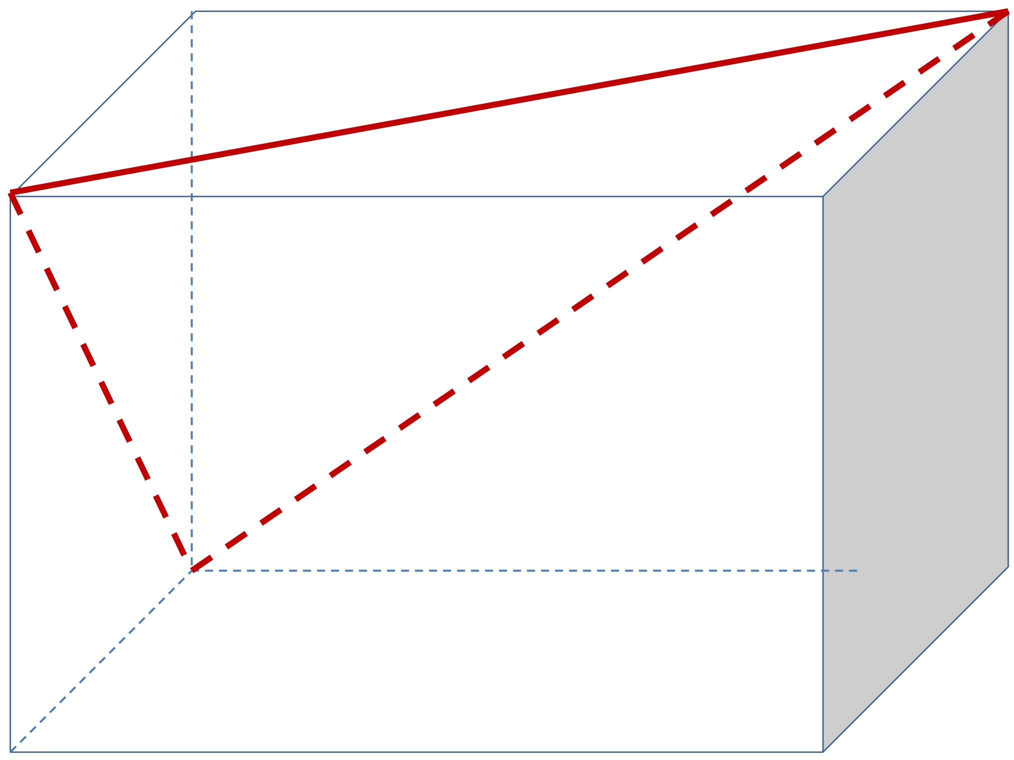
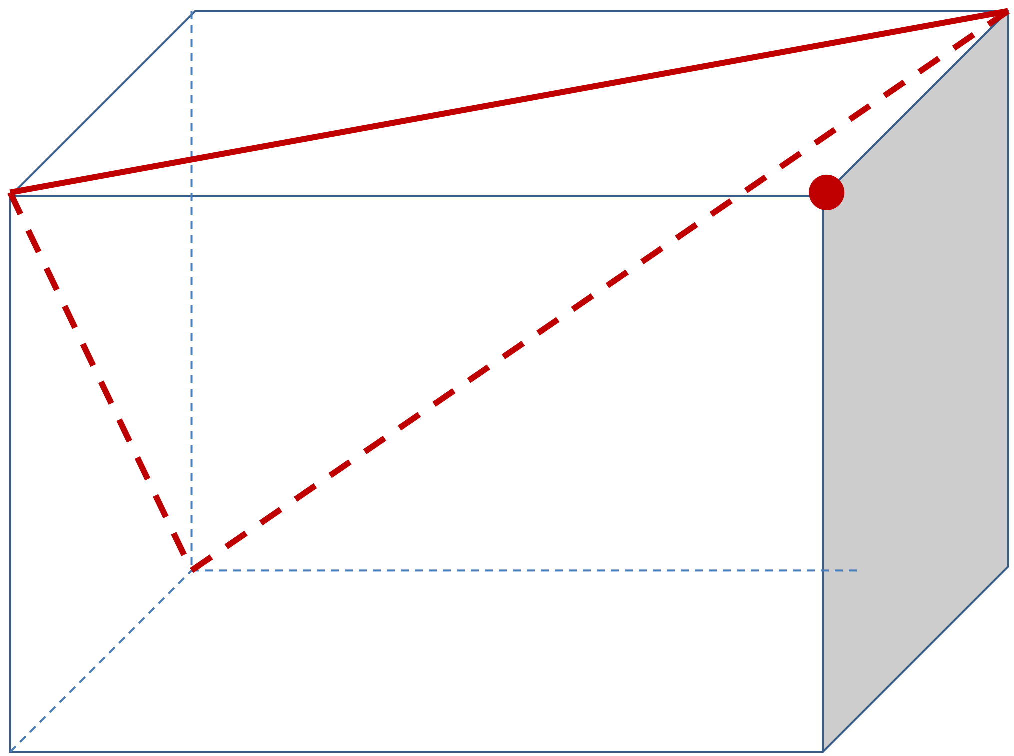
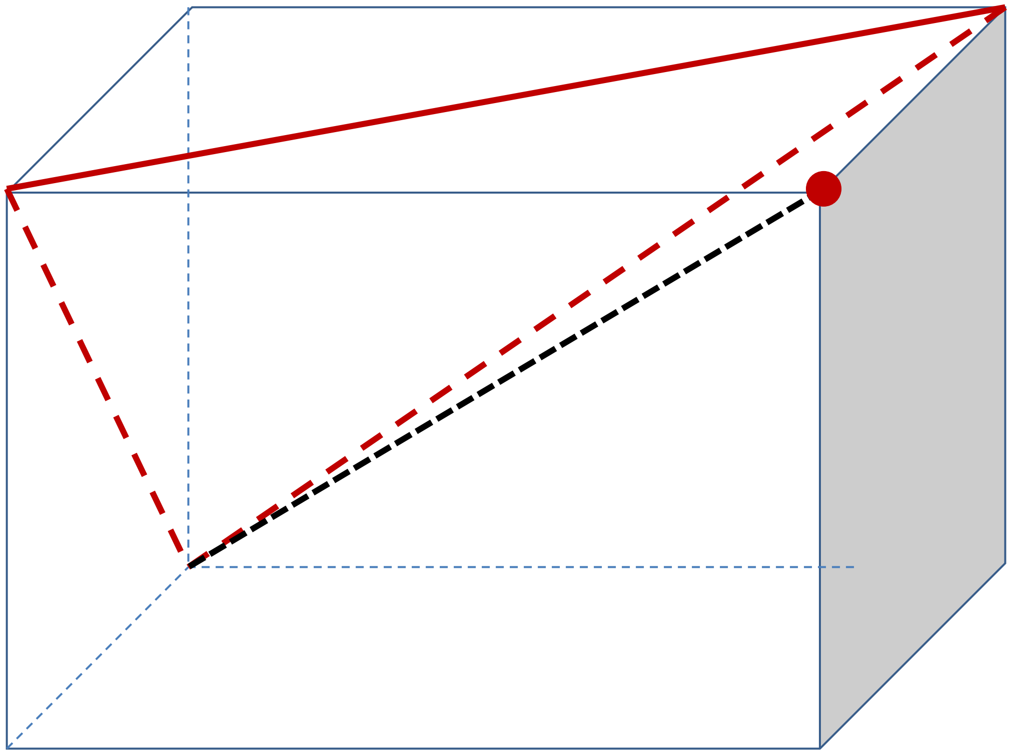
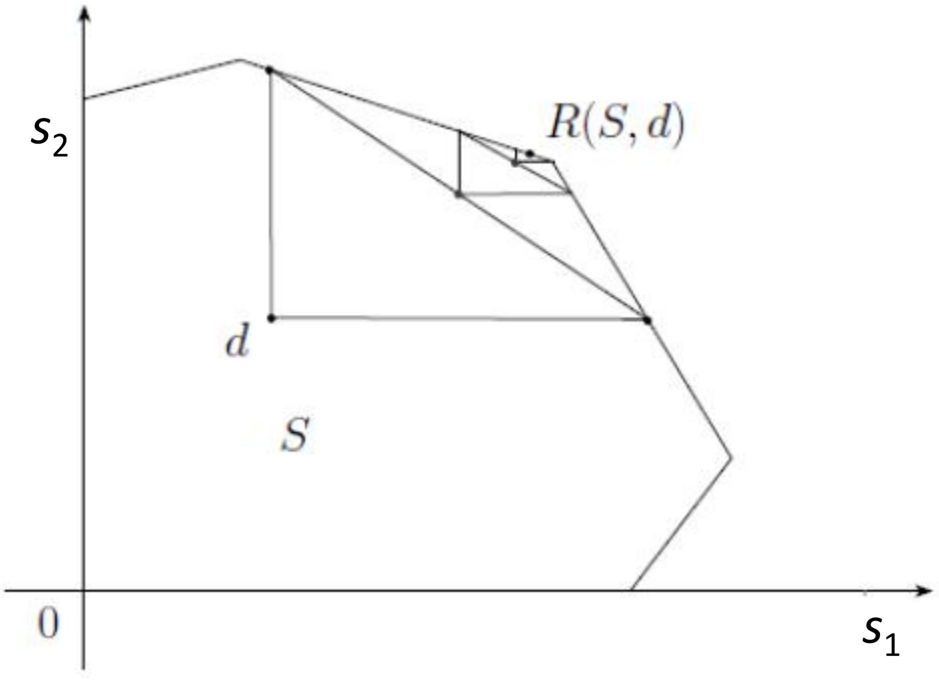
4. *Individual monotonocity.* If *d* is in *S* and $S \subset T$ and the utopian point is the same for both (*S, d*) and (*T, d*), then *f*(*T, d*) = *f*(*S, d*), provided *f*(*T, d*) is a Pareto efficient point of *S.*







The Discrete Raiffa Solution



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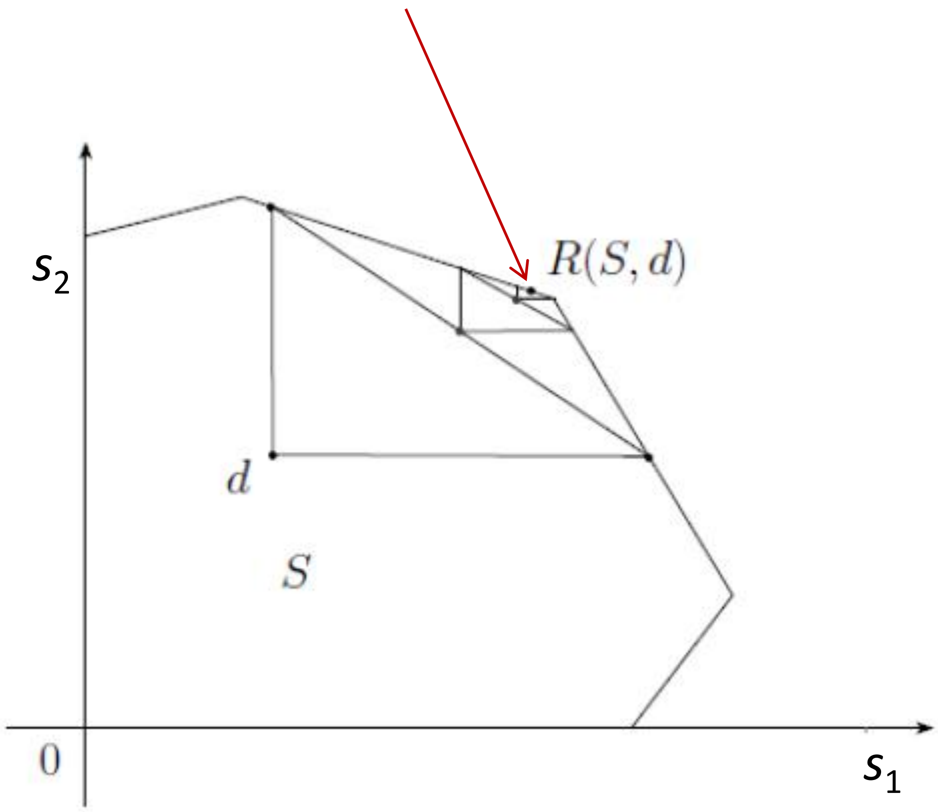

The discrete Raiffa solution

The *discrete Raiffa solution* is defined as the limit of a sequence $\{x^k\}$ of points from S generated as follows:

Let (S, d) be an instance of $\mathcal{B}$ and let $m_i(S, x)$ denote the maximum of the function

$$(y_1, y_2) \mapsto y_i$$

over the individually rational part of (S, x), that is, over the set

$$S_x^+ = \{(y_1, y_2) \in S : x_1 \leq y_1, x_2 \leq y_2\}.$$

Set $x^0 = d$, and continue inductively by defining x^{k+1} as the middle point of the line segment connecting the points

$$(x_1^k, m_2(S, x^k)) \quad \text{and} \quad (m_1(S, x^k), x_2^k).$$

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3. ***Ordinality*****. For each instance (*****S, d*****), we have: If *****A***** is an order-preserving transformation, then *****f*****(*****A*****(*****S*****), *****A*****(*****d*****)) = *****A*****(*****f*****(*****S, d*****)).**

Ordinal solutions

There are no interesting ordinal solutions for Nash's two player problem $\mathcal{B}$. Consider the instance $(S, 0)$ with

$$S = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1\}.$$

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Let T be the transformation defined by

$$T(x_1, x_2) = \left(\frac{2x_1}{1 + x_1}, \frac{x_2}{2 - x_2} \right).$$

It can easily be verified that T preserves utility orderings of both players on the unit square

$$Q = \{(x_1, x_2) : 0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq 1\},$$

and that it maps the set S onto itself.

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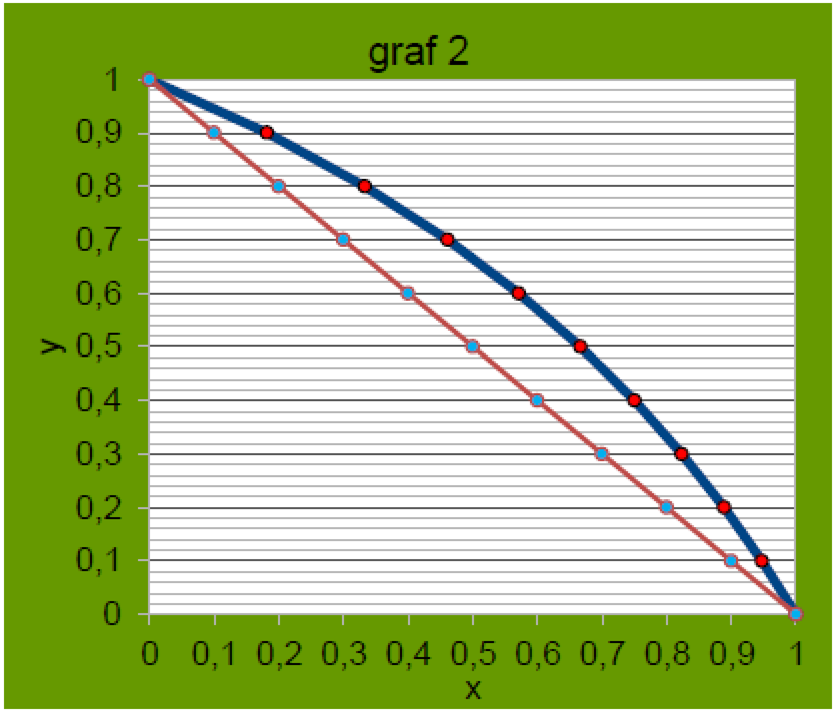

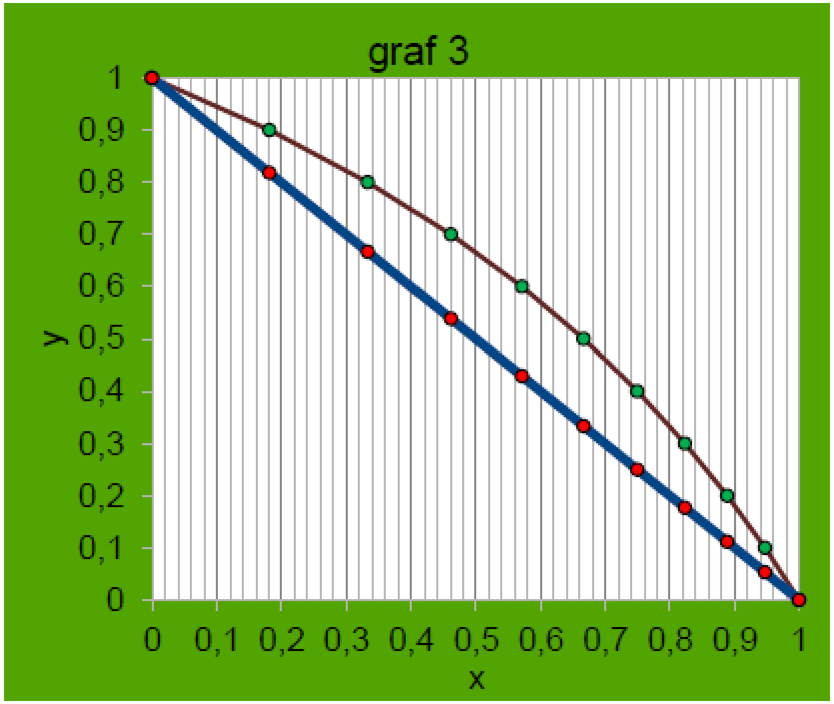

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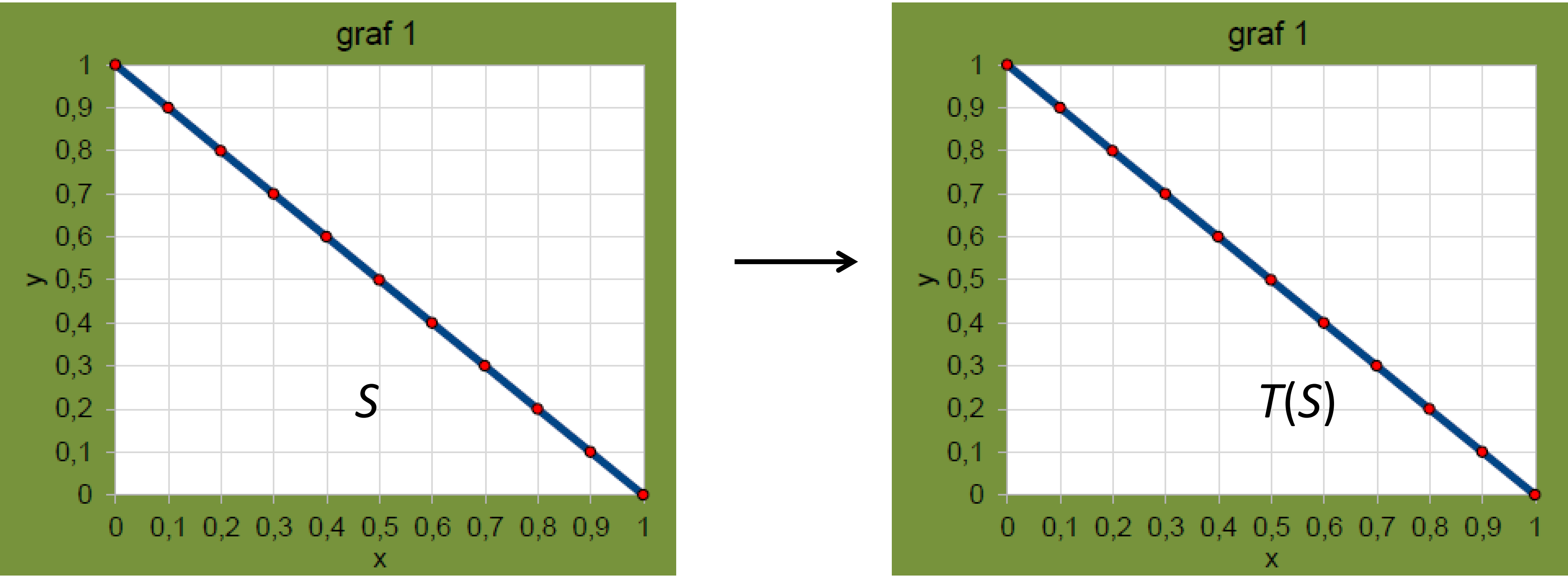


It follows that each solution f to $\mathcal{B}$ which is invariant with respect to ordinal transformations must assign to the instance $(S, 0)$ a point in S which is also a fixed point (on Q) of mapping

$$T(x_1, x_2) = \left(\frac{2x_1}{1 + x_1}, \ \frac{x_2}{2 - x_2}\right).$$

However, the only fixed points of T on Q are

$$(0,0), (0,1), (1,0), (1,1).$$

The point $(1,1)$ is infeasible because it does not belong to S, and the remaining points are uninteresting: $(0,0)$ is the point of disagreement, and points $(0,1)$ $(1,0)$, are so called dictatorial solutions.

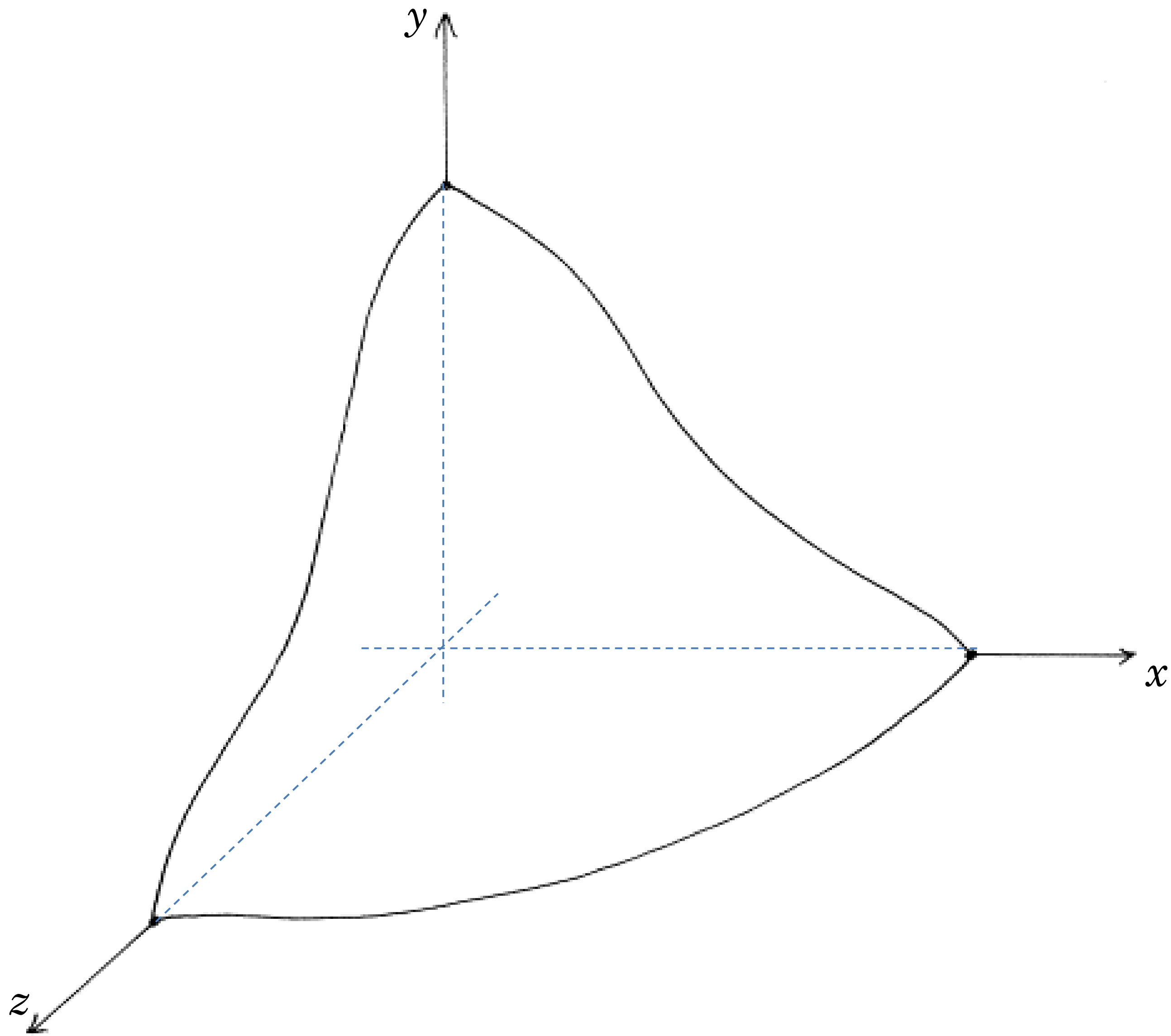
The main argument of this proof cannot be extended to Nash's bargaining problem with three or more players, and non-dictatorial ordinally invariant solution exist for problems with more than two players.

One such solution has been proposed by Shapley and Shubik. The construction is based on the following observation.

If $\mathcal{P}$ is a Pareto surface in $\mathcal{R}^3$ such that every unbounded curve in $\mathcal{R}^3$ (beginning in d and moving weakly monotonically) meets $\mathcal{P}$ in precisely one point, and (a_1, a_2, a_3) is a point in $\mathcal{R}^3 \backslash \mathcal{P}$, then there is a unique point (b_1, b_2, b_3) such that the points

$$(a_1, b_2, b_3), (b_1, a_2, b_3), (b_1, b_2, a_3)$$

belong to $\mathcal{P}$.


y
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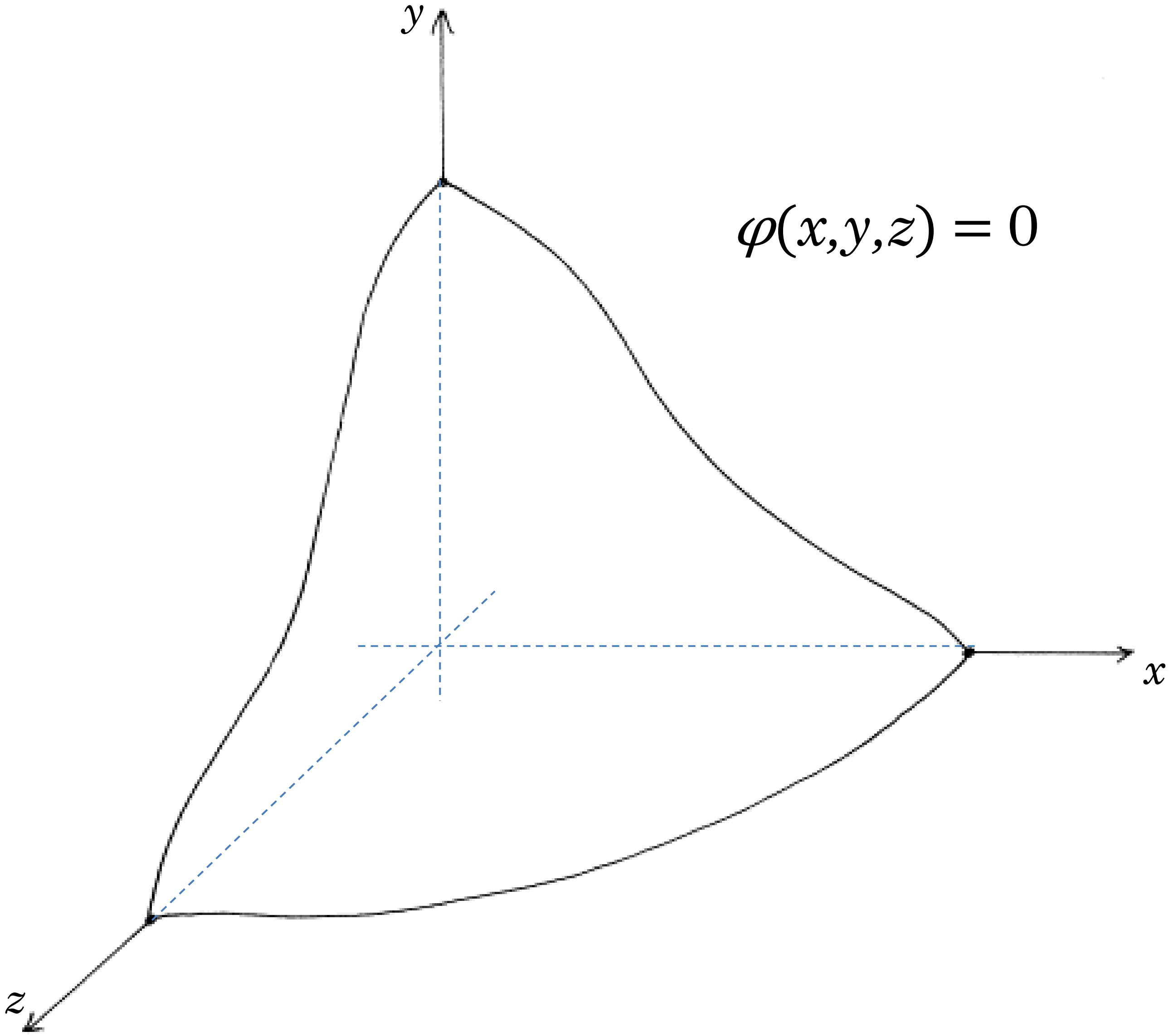
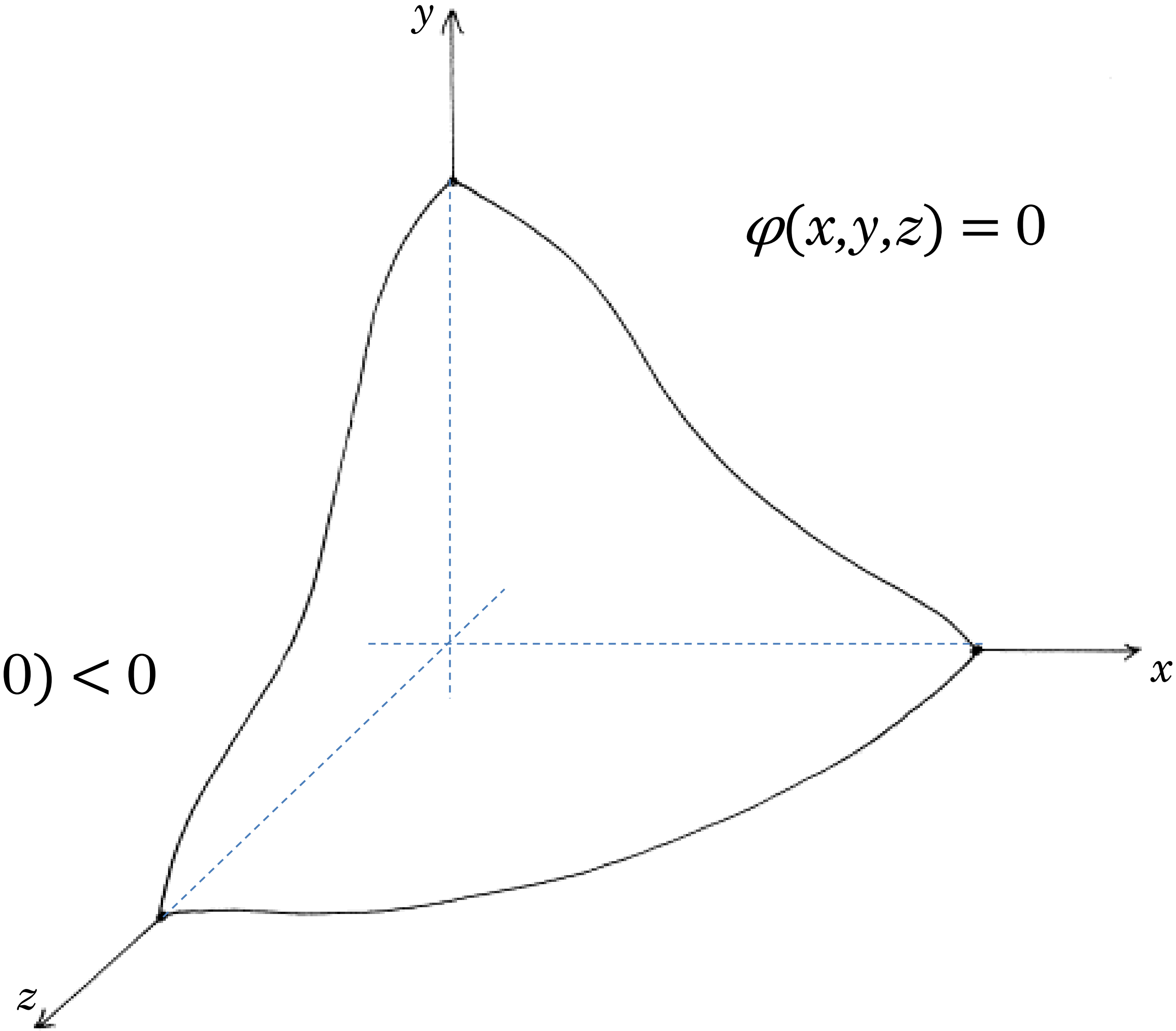


$$\varphi(x,y,z) = 0$$



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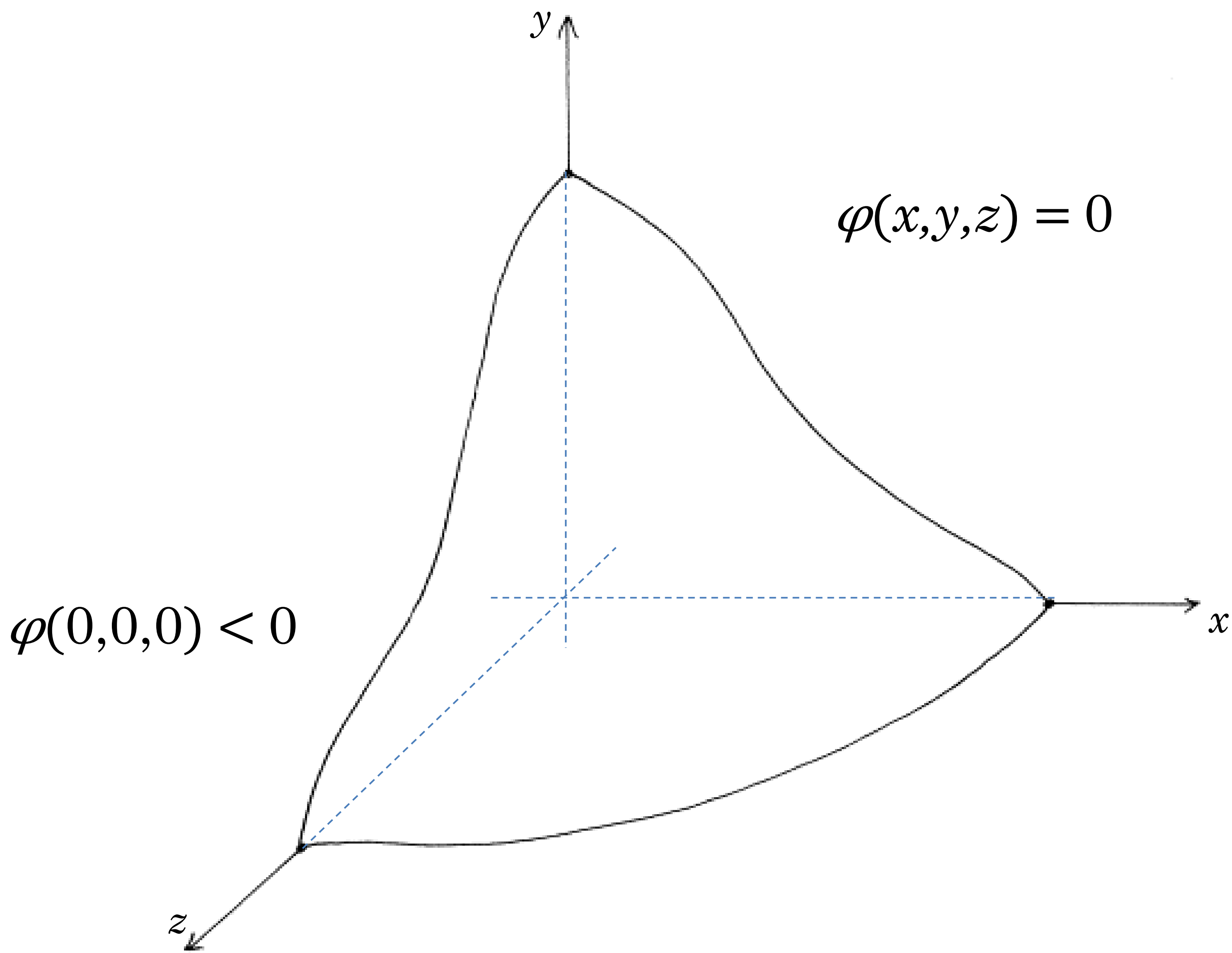
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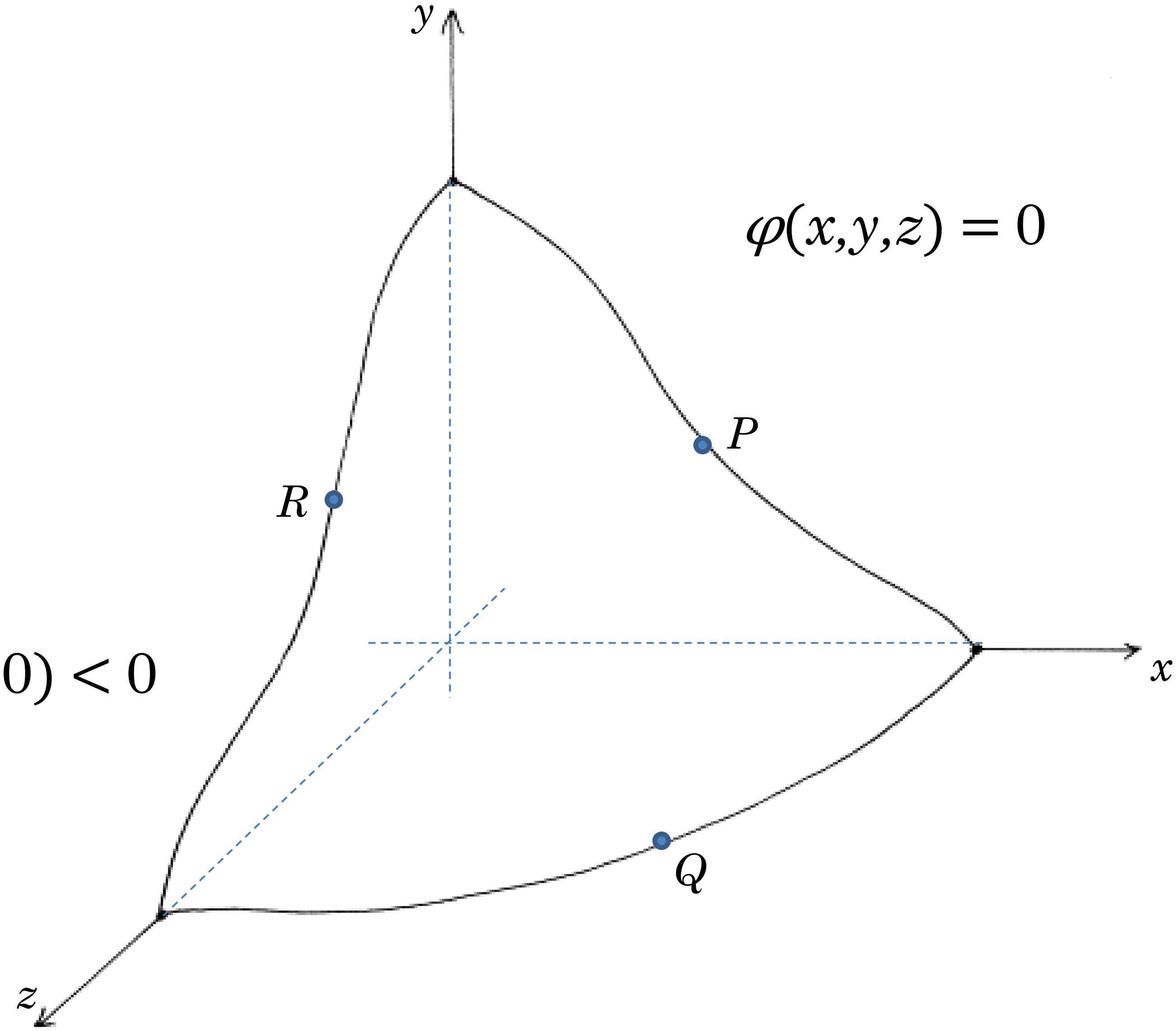

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Using this fact, one can define the Shapley-Shubik solution for an instance (S, d) as the limit of sequence $\{x^k\}_1^\infty$ of points defined by setting

$$x^0 = (x_1^0, x_2^0, x_3^0) \quad \text{with} \quad (x_1^0, x_2^0, x_3^0) = (d_1, d_2, d_3),$$

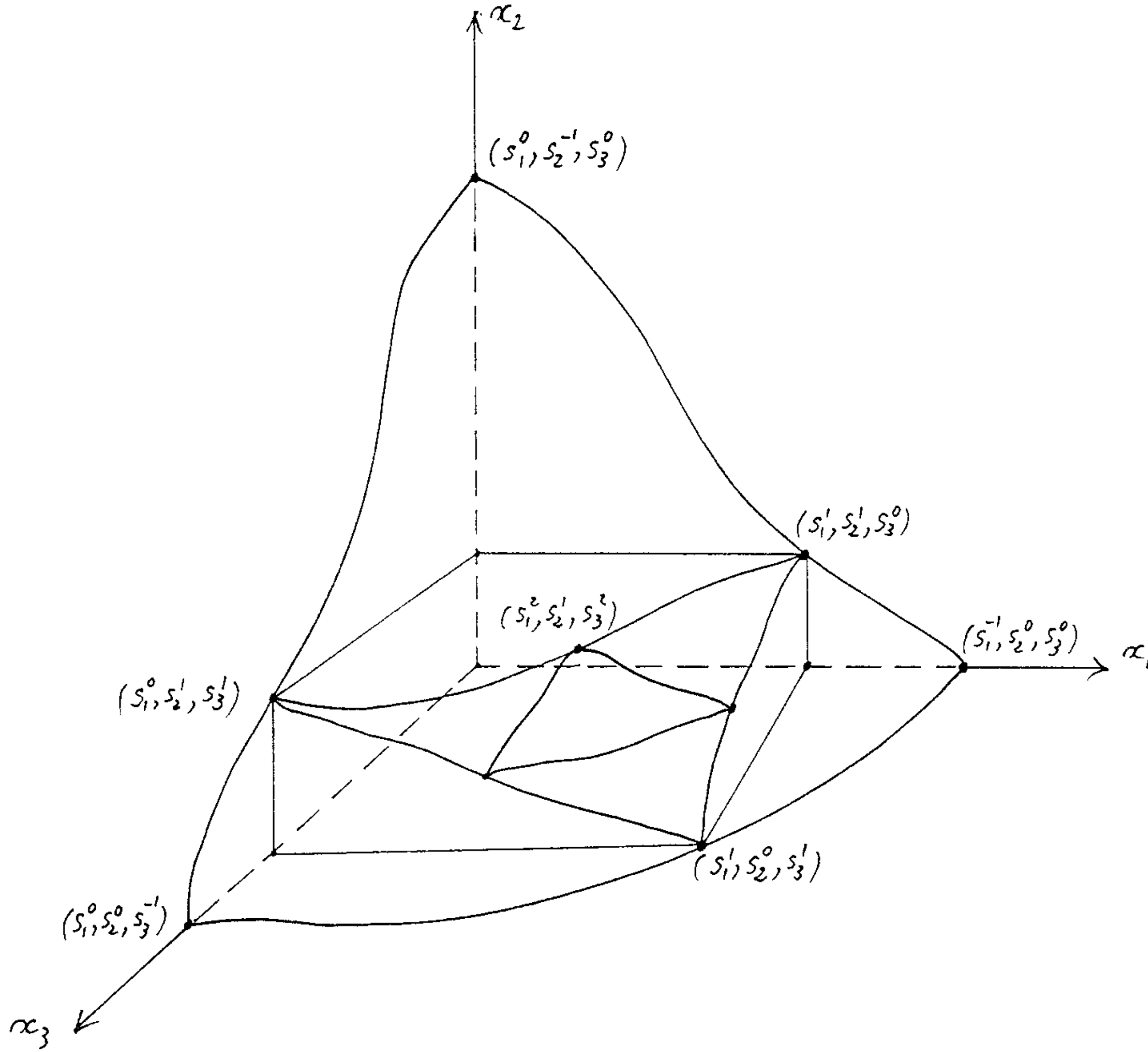
and defining x^k to be the unique point

$$x^k = (x_1^k, x_2^k, x_3^k)$$

determined by the property that the points

$$(x_1^{k-1}, x_2^k, x_3^k), (x_1^k, x_2^{k-1}, x_3^k), (x_1^k, x_2^k, x_3^{k-1})$$

belong to the Pareto surface of S.


x_2
(s_1^0, s_2^{-1}, s_3^0)
(s_1^1, s_2^1, s_3^0)
(s_1^2, s_2^1, s_3^2)
(s_1^{-1}, s_2^0, s_3^0)
x_1
(s_1^0, s_2^1, s_3^1)
(s_1^1, s_2^0, s_3^1)
(s_1^0, s_2^0, s_3^{-1})
x_3

An alternative solution

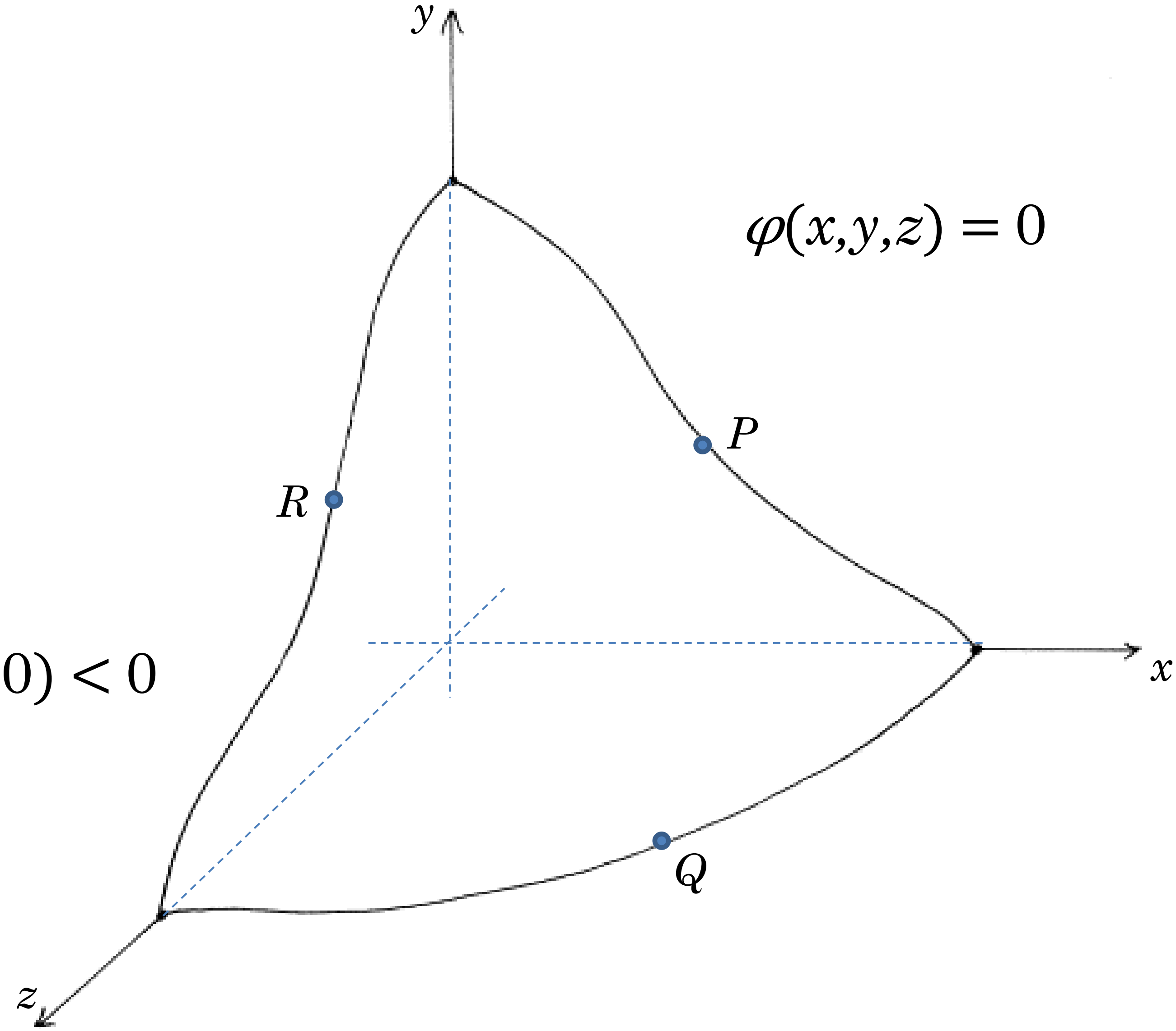
We propose to combine the ideas of the Shapley-Shubik solution and the Discrete Raiffa solution to obtain a solution to an instance (S, d) from $\mathcal{B}$ as the limit of the sequence $\{y^k\}$ of points from S. Again, we set $y^0 = (d_1, d_2, d_3)$. Let (x_1, x_2, x_3) be the point obtained from y^0 by one step of the Shapley-Shubik procedure. We construct the next point y^1 by the same averaging that is used in the Sequential Raiffa procedure, but now using the points

$$(y_1^0, x_2, x_3),\ (x_1, y_2^0, x_3),\ (x_1, x_2, y_3^0)$$

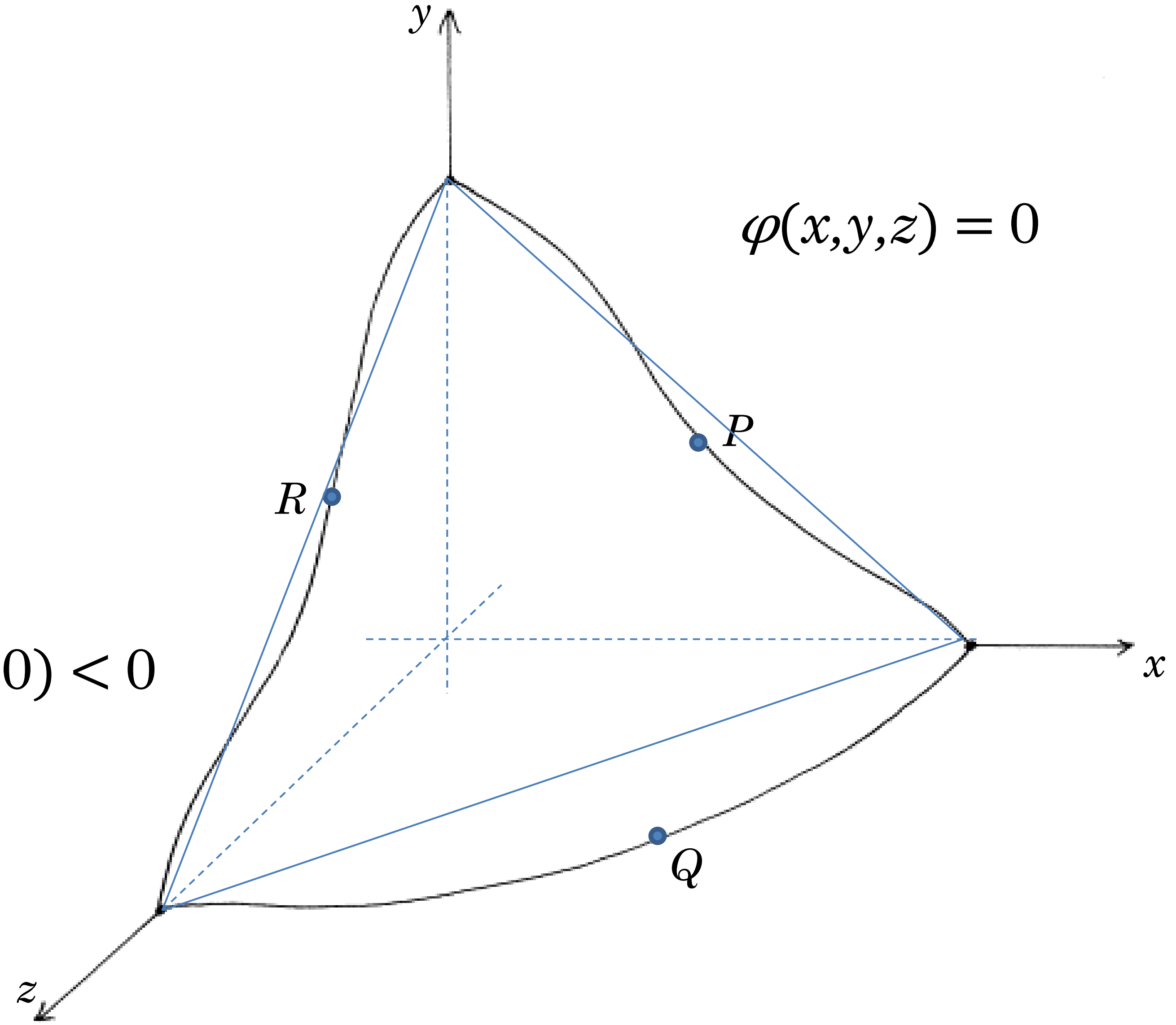
instead of using the points

$$(m_1(S, y^0), y_2^0, y_3^0),\ (y_1^0, m_2(S, y^0), y_3^0),\ (y_1^0, y_2^0, m_3(S, y^0)).$$

$$\varphi(x,y,z) = 0$$

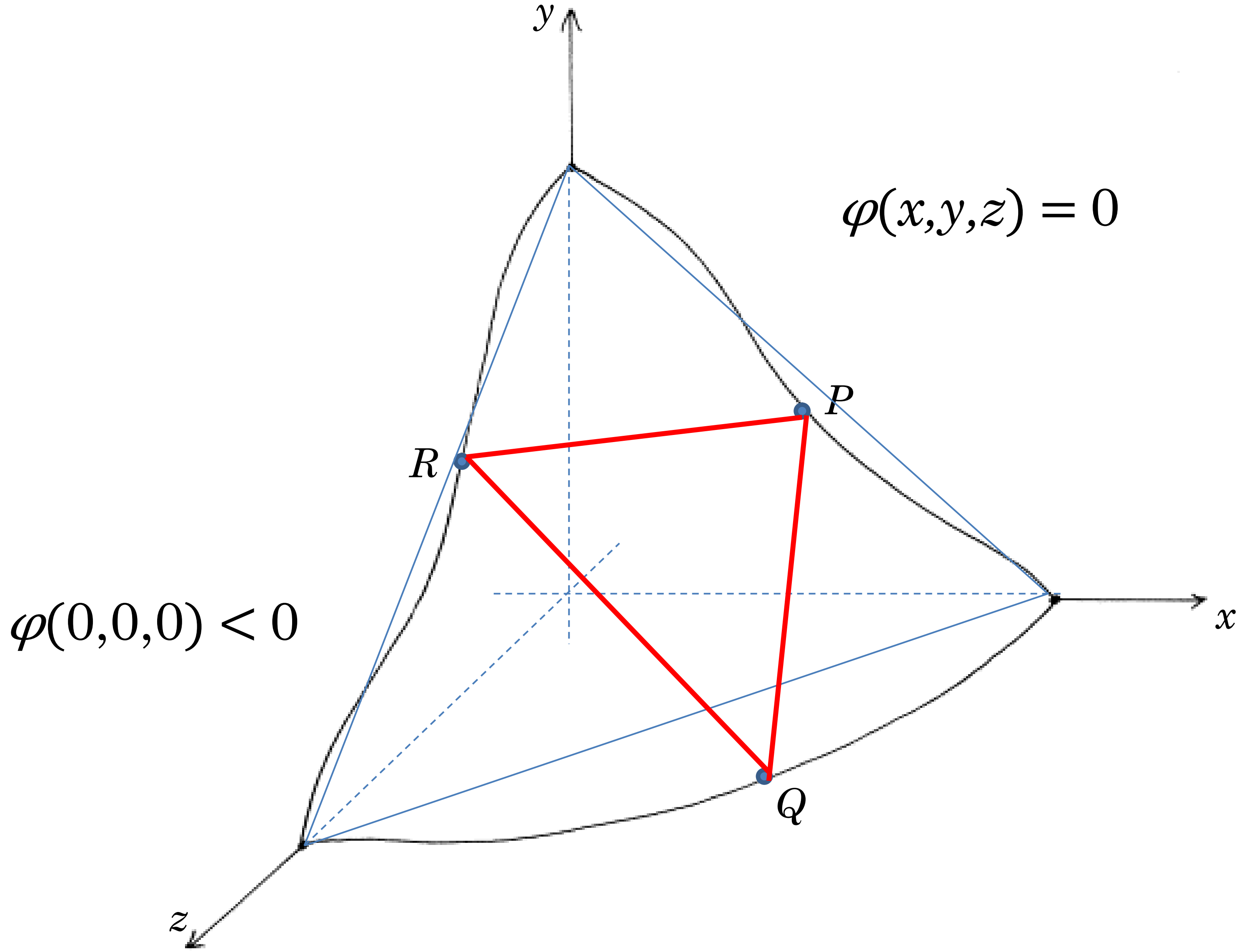


$$\varphi(0,0,0) < 0$$



$$\varphi(x,y,z) = 0$$

$$\varphi(0,0,0) < 0$$


y
$\varphi(x,y,z) = 0$
P
R
$\varphi(0,0,0) < 0$
x
Q
z

The procedure seem to be new. It can be used not only for instances from $\mathcal{B}$. For example, it can be applied to the three player problems to which it is possible to apply the Shapley-Shubik procedure, provided that the results of the averaging belong to the bargaining set.

It would be interesting to compare the procedure with other available procedures on some standard classes of problems.

Extensions to problems with more than three players would also be of interest.

However, as the main open question we consider the problem of establishing systems of axioms that define the proposed solution uniquely on reasonable classes of bargaining problems.

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Everything has been thought of before, but the problem is to think of it again.

Goethe

Step-by-step solutions

Recently Diskin et al. have provided an axiomatization of a family of generalized Raiffa's solutions for cooperative bargaining.

They propose a solution concept which is composed of two solution functions. One solution function specifies an interim agreement and the other specifies the terminal agreement. Such a step-by-step solution concept can formally be defined as follows.

A pair (f, g) of functions from $\mathcal{B}$ into $\mathcal{R}^n$ is called the *step-wise solution* if both $f(S, d)$ and $g(S, d)$ belong to S for each instance (S, d) of $\mathcal{B}$. The first component f specifies the interim agreement and the second component g specifies the terminal agreement.

Generalized Raiffa's solutions

The *set of generalized Raiffa solutions* is a family of step-wise bargaining solutions $\{(f^p, g^p)\}_{0<p\leq 1}$ where f^p and g^p are defined in the following way. For $0 < p \leq 1$, the function f^p is defined by

$$f^p(S,d) = d + \frac{p}{n}(U(S,d) - d),$$

and the function g^p is defined by

$$g^p(S,d) = d^\infty(S,d),$$

where $d^\infty(S,d)$ is the limit of the sequence $\{d^k(S,d)\}$ of points constructed inductively by

$$d^0(S,d) = d \text{ and } d^{k+1}(S,d) = f^p(S,d^k).$$

Axiom 1. $g(S, d) = g(S, f(S, d))$.

Axiom 2. $g(S, d)$ is individually rational.

Axiom 3. If $f(S, d)$ is individually rational, and if d is not Pareto optimal in S, then $f(S, d) \neq d$.

Axiom 4. If all players are symmetric in (S, d), then they are also symmetric in $f(S, d)$.

Axiom 5. $f(A(S), A(d)) = A(f(S, d))$.

Axiom 6. If $S \subset T$, then $f(S, d) \leq f(T, d)$.

Axiom 7. If $S_d = T_d$, then $f(S, d) = f(T, d)$.

Offical Answers

- The problem under consideration appears almost everywhere in Nature.
- It represents an important issue associated with many real world problems.
- Many interesting instances of the problem can be handled by our approach.

Unoffical Answers

- Who cares.
- Whatever the origin of the problem might be, the problem itself is interesting and challenging.
- The problem was created as a by-product of our previous research.