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Incorrectly Posed Systems of (max, min)-linear Equations and Inequalities.

Karel Zimmermann
(Mahmoud Gad)

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CONTENT:

- Notations
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Basic Concepts.

- Max-separable function $f : R^n \longmapsto R$:

$$f(x) = \max_{j \in J} f_j(x_j),$$

where $J = 1, \dots, n,\ x = (x_1, \dots, x_n)$.

- Special cases:
(max, +)-linear functions

$$f(x) = \max_{j \in J}(c_j + x_j),$$

(max, min)-linear functions

$$f(x) = \max_{j \in J}(\min(c_j, x_j)) = \max_{j \in J}(c_j \wedge x_j).$$

Basic Concepts.

- System of max-separable inequalities:

$$\max_{j\in J}(a_{ij} \wedge r_{ij}(x_j)) \geq b_i,\ \ i \in I, \tag{1}$$

$$\max_{j\in J}(a_{ij} \wedge r_{ij}(x_j)) \leq b_i,\ \ i \in K, \tag{2}$$

$$\underline{x}_j \leq x_j \leq \overline{x}_j,\ j \in J, \tag{3}$$

where $a_{ij},\ , b_i,\ \underline{x}_j,\ \overline{x}_j \in R,\ \ r_{ij} : R \longmapsto R$, Range$(r_{ij}) = R$, continuous and strictly increasing functions, $I,\ K$ finite index sets, $\alpha \wedge \beta \equiv \min(\alpha, \beta)$ for $\alpha, \beta \in R$.

- If $r_{ij}(x_j) = d_{ij} + x_j$ and a_{ij} sufficiently large, we obtain a $(\max, +)$-linear system.
- If $r_{ij}(x_j) = x_j$, we obtaining a $(\max, \min)-$linear system.

- [Vorobjov], [Korbut], [Carre], [Cuninghame-Green], [Gondran], [Minoux], [Helbig], [Nachtigal], [Olsder], [Maslov], [Litvinov], [Krivulin], [Pap], [Gaubert], [Akian], [de la Ponte], [Sergeyev], [Nitica], [Singer], [Nedoma], [Butkovic], [Gavalec], [Cechlarova], and others.
  $(R, \oplus, \otimes) = (\max, +)$, $(R, \oplus, \otimes) = (\max, \min)$ and others

$$f(x) = \Sigma_j^{\oplus}(c_j \otimes x_j) = c_1 \otimes x_1 \ \oplus \ \dots \ c_n \otimes x_n$$

  $(\oplus, \otimes)$-linear functions.

- Extremal algebra, path algebra, max-min algebra, max-plus algebra, fuzzy algebra, idempotent algebra, tropical algebra etc

- Applications to machine time scheduling, capacity and reliability of networks, discrete event problems, fuzzy set problems.

Motivating Example 1

- **Example 1.**
- a_{ij}............................capacity of link $D_i P_j$;
- x_j...............capacity of link $P_j T$ (to be found);
- $a_{ij} \wedge x_j$..........capacity of $D_i P_j T$;
- $a_i(x) = \max_{j \in J} = (a_{ij} \wedge x_j)$;
- Requirements:
  $a_i(x) = b_i, \ \forall i \in I$ or $\underline{b}_i \leq a_i(x) \leq \overline{b}_i \ \ \forall i \in I$

Motivating Example 2

- **Example 2.**
- Let us assume that we have m fuzzy sets A_i, $i \in I \equiv \{1, \ldots, m\}$ with a finite support $J \equiv \{1, \ldots, n\}$ and membership functions $\mu_i : J \to [0, 1]$. We have to find fuzzy set X with membership function $\mu_X : J \to [0, 1]$. Let functions $\mu_{iX} : J \to [0, 1]$ be defined as follows:
- $$\mu_{iX}(j) \equiv \mu_i(j) \wedge \mu_X(j),$$

  where symbol $\wedge$ is used to denote the minimum of two numbers, i.e. $\alpha \wedge \beta \equiv \min(\alpha, \beta)$ for any real numbers α, β. Then for each $i \in I$ function μ_{iX} is the membership function of the intersection of fuzzy set A_i and X.

Motivating Example 2 - conitnued.

- The expressions

$$H_i(X) \equiv \max_{j \in J}(\mu_{iX}(j))$$

  are the heights of the intersections of fuzzy sets A_i, X for all $i \in I$.
- We require that the heights $H_i(X)$ are equal $\hat{b}_i$ for all $i \in I$, i.e.
- $$H_i(X) \equiv \max_{j \in J}(\mu_{iX}(j)) = \hat{b}_i, \ \forall i \in I. \qquad (*)$$

Motivating Example 2 - conitnued.

- Let us set $a_{ij} \equiv \mu_i(j),\ x_j \equiv \mu_X(j)$ for all $i \in I,\ j \in J$.
- Then in this new notations relations (*) have the form
- $$\max_{j \in J}(a_{ij} \wedge x_j) = \hat{b}_i, \forall i \in I,$$

  which is the system of $(\max, \min)$-linear equations.

Properties of the Inequality System.

- We can replace the inequality system (2) - (3) by
- $\underline{x}_j \leq x_j \leq x_j(b) \ \ \forall j \in J$, where

  $x_j(b) = \min_{i \in I_j} r_{ij}^{-1}(b_i) \wedge \overline{x}_j, \forall j \in J$ and $I_j = \{i \in K; a_{ij} > b_i\}$
- If x solves the inequality system (1) - (3), it must be $\underline{x} \leq x \leq x(b)$

Properties of the Inequality System.

- We replace system (1) - (3) by
- $$\max_{j \in J}(a_{ij} \wedge r_{ij}(x_j)) \geq b_i, \; i \in I \tag{4}$$
  $$\underline{x} \leq x \leq x(b) \tag{5}$$
- Let $M(b)$ denote the set of solutions of system (4) - (5).
- If $M(b) \neq \emptyset$, then $\underline{x} \leq x(b)$.

Properties of the Inequality System.

- Let for all $i \in I,\ j \in J$

$$T_{ij} = \{x_j\ ;\ \underline{x}_j \leq x_j \leq x_j(b)\ \&\ a_{ij} \wedge r_{ij}(x_j) \geq b_i\}$$

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- **Lemma 1.**
  If $T_{ij} \neq \emptyset$, then

$$T_{ij} = [\max(\underline{x}_j, r_{ij}^{-1}(b_i)), x_j(b)]$$

.

- **Lemma 2.**
  Let $j \in J$ be fixed, $I = \{1,\ \ldots,\ m\}$. Then there exists a permutation $\{i_1,\ \ldots,\ i_m\}$ of indices of I such that

$$T_{i_1 j} \subseteq T_{i2j} \subseteq\ \ldots\ T_{i_m j}.$$

Properties of the Inequality System.

- **Lemma 3.**
  $M(b) \neq \emptyset$ if and only if $\forall i \in I \; \exists j(i) \in J$ such that $T_{ij(i)} \neq \emptyset$.

Incorrectly Posed Problems - Formulation.

- Let us consider the following system of equations:
- $$a_i(x) \equiv \max_{j \in J}(a_{ij} \wedge x_j) = \hat{b}_i, \; i \in I \tag{8}$$

  where $a_{ij}, \; \hat{b}_i \in R, \forall i \in I \equiv \{1, \; \ldots, \; m\}, j \in J \equiv \{1, \; \ldots, \; n\}$.
- Let the set of solutions of system (8) be denoted $M(\hat{b})$.
- Let $M(\hat{b}) = \emptyset$.

Incorrectly Posed Problems - Formulation.

- We will consider the following optimization problem:
-

$$\left\| b - \hat{b} \right\| \equiv \max_{i \in I} \left| b_i - \hat{b}_i \right| \longmapsto \min \tag{9}$$

subject to

$$b \in R(A) \equiv \{ b \in R^m \ ; \ M(b) \neq \emptyset \} \tag{10}$$

- Note that

$$R(A) = \{ b \ ; \ \exists x \in R^n \ \texttt{such} \quad \texttt{that} \ a_i(x) = b_i, \ \forall i \in I \}.$$

and

$$R(A) = Range(A : R^n \longmapsto R^m),$$

where we set $A(x) = (a_1(x), \ \ldots, \ a_m(x))$.

Incorrectly Posed Problems - Reformulation.

- Let for any $t \in R$

$$M(\hat{b}, t) \equiv \{b \in R(A)\ ;\ \left\| b - \hat{b} \right\| \leq t\}.$$

- Let us consider the following optimization problem:

$$t \longmapsto \min \tag{11}$$

subject to

$$M(\hat{b}, t) \neq \emptyset \tag{12}$$

- Note that

$$M(\hat{b}, t) = \{b = A(x); \exists x \in R^n \texttt{ such that } \hat{b}_i - t \leq a_i(x) \leq \hat{b}_i + t \forall i\}$$

Incorrectly Posed Problems - Reformulation.

- In other words $M(\hat{b}, t) \neq \emptyset$ if and only if inequality system

$$\hat{b}_i - t \leq a_i(x)\, \forall i \in I \;\&\; a_{ij} \wedge x_j \leq \hat{b}_i + t,\ \forall i \in I, \forall j \in J \tag{13}$$

  has a solution.

- Therefore $M(\hat{b}, t)$ is non-empty if system (13) is solvable. Then problem (11) – (12) is equivalent to

- $$t \longmapsto \min \tag{14}$$

  subject to (13) has a solution (15)

Incorrectly Posed Problems - Reformulation.

- Let $\hat{b} + t = (\hat{b}_1 + t, \ \ldots, \ \hat{b}_m + t)$.
- Note that $a_{ij} \wedge x_j \leq \hat{b}_i + t \ \forall i \in I, \ j \in J$ implies
- $$x_j \leq x_j(\hat{b} + t) \ \ \forall j \in J,$$
- where for all $j \in J$

$$x_j(\hat{b} + t) \equiv \min_{k \in I_j(t)} \hat{b}_k + t, \ I_j(t) = \{k \in I \ ; \ a_{kj} > \hat{b}_k + t\}.$$

Incorrectly Posed Problems - Reformulation.

- it follows that $b = A(x) \in M(\hat{b}, t)$ implies $x \leq x(\hat{b} + t)$.
- Let for all $i \in I,\ j \in J$

$$T_{ij}(t) = \{x_j \ ;\ \hat{b}_i - t \leq a_{ij} \wedge x_j \ \&\ x_j \leq x_j(\hat{b} + t)\},$$

- Note that
  (a) $T_{ij}(t) \neq \emptyset$ for a sufficiently large t;
  (b) $\hat{b}_i - t$ is strictly decreasing in t and $x_j(\hat{b} + t)$ is partially continuous and strictly increasing in t with maximum m jumps.

Incorrectly Posed Problems - Reformulation.

- **Lemma 4.**
  $T_{ij}(t) \neq \emptyset$ if and only if $\hat{b}_i - t \leq a_{ij} \wedge x_j(\hat{b} + t)$.

- **Lemma 5.**
  For any $i \in I,\ j \in J$ there exists τ_{ij} such that $T_{ij}(t) \neq \emptyset$ if and only if $t \geq \tau_{ij}$.

- **Theorem 2.**
  Let t^{opt} be the optimal solution of optimization problem $(14) - (15)$ and b^{opt} be the optimal solution of optimization problem $(9) - (10)$. Then we have:

$$t^{opt} = \max_{i \in I} \min_{j \in J} \tau_{ij},$$

$$b^{opt} = A(x(\hat{b} + t^{opt}))$$

Modifications of the Problem.

- (1) we can consider an unsolvable system of inequalities;
- (2) Additional restrictions on $b \in R(A)$, e.g. $b_i = \hat{b}_i$ for some $i \in I$;
- (3) Changing of a_{ij}'s instead of the right hand sides $\hat{A} - t \leq A \leq \hat{A} + t$;
- (4) Other max-separable objective functions of the form $f(b) = \max_{i \in I} f_i(b_i)$ defined on $R(A)$, e.g. $\max_{i \in I} w_i \left| b_i - \hat{b}_i \right|$.

Generalizations or Extensions of the Problem.

- We can consider systems of the form

$$\max_{j \in J}(a_{ij} \wedge r_{ij}(x_j) = \hat{b}_i, \ i \in I,$$

where $r_{ij} : R \longmapsto R$ are strictly increasing and continuous.

- We can consider various formulations of the problem for some types of two-sided systems.
- Post optimal analysis of the problems (i.e. finding out which changes will improve the value of the objective function).
- Interval coefficients.